

# GLOBAL SOLUTIONS TO THE OLDROYD-B MODEL WITH A CLASS OF LARGE INITIAL DATA

DAOYUAN FANG, RUIZHAO ZI

**ABSTRACT.** Consider a global wellposed problem for the incompressible Oldroyd-B model. It is shown that this set of equations admits a unique global solution provided the initial horizontal velocity  $u_0^h$ , the product  $\omega u_0^d$  of the coupling parameter  $\omega$  and initial the vertical velocity  $u_0^d$ , and initial symmetric tensor of constrains  $\tau_0$  are sufficient small in the scaling invariant Besov space  $\dot{B}_{2,1}^{\frac{d}{2}-1} \times \dot{B}_{2,1}^{\frac{d}{2}}$ ,  $d \geq 2$ . In particular, the result implies the global well-posedness of Oldroyd-B model with large initial vertical velocity  $u_0^d$ .

## 1. INTRODUCTION

In this paper, we focus on a prototypical model for viscoelastic fluids, the so called Oldroyd-B model, which was originally introduced by J.G. Oldroyd [28]. This type of fluids is described by the following set of equations

$$(1.1) \quad \begin{cases} \operatorname{Re}(u_t + (u \cdot \nabla)u) - (1 - \omega)\Delta u + \nabla \Pi = \operatorname{div} \tau, \\ \operatorname{We}(\tau_t + (u \cdot \nabla)\tau + g_\alpha(\tau, \nabla u)) + \tau = 2\omega D(u), \\ \operatorname{div} u = 0, \end{cases}$$

where  $u$  is the velocity of the fluid,  $\tau$  is the symmetric tensor of constrains, and  $\Pi$  is the pressure which can be regarded as the Lagrange multiplier for the divergence free condition.  $\operatorname{Re}$  and  $\operatorname{We}$  denote the Reynolds and Weissenberg number of the fluid, respectively.  $\omega \in (0, 1)$  is the coupling parameter. Moreover, the quadratic term  $g_\alpha$  is given by  $g_\alpha(\tau, \nabla u) := \tau W(u) - W(u)\tau - \alpha(D(u)\tau + \tau D(u))$  for some  $\alpha \in [-1, 1]$ , where  $D(u) = 1/2(\nabla u + (\nabla u)^\top)$  is the deformation tensor and  $W(u) = 1/2(\nabla u - (\nabla u)^\top)$  is the vorticity tensor.

A brief derivation of system (1.1) is as follows. In fact, the dynamics of homogeneous, isothermal, and incompressible fluid is described by

$$(1.2) \quad \begin{cases} \partial_t u + (u \cdot \nabla)u = \operatorname{div} \mathbb{T}, \\ \operatorname{div} u = 0, \end{cases}$$

where  $u, \mathbb{T}$  are the velocity and stress tensor, respectively. Moreover,  $\mathbb{T}$  can be decomposed into  $-p \operatorname{Id} + \tau$ , where  $p$  denotes the pressure of the fluid and  $\tau$  the tangential part of the stress tensor, respectively. For the Oldroyd-B model,  $\tau$  is given by the relation

$$(1.3) \quad \tau + \lambda \frac{\mathcal{D}_\alpha \tau}{\mathcal{D}t} = 2\eta \left( D(u) + \mu \frac{\mathcal{D}_\alpha D(u)}{\mathcal{D}t} \right),$$

where  $\frac{\mathcal{D}_\alpha}{\mathcal{D}t}$  denotes the “objective derivative”

$$\frac{\mathcal{D}_\alpha \tau}{\mathcal{D}t} = \partial_t \tau + (u \cdot \nabla)\tau + g_\alpha(\tau, \nabla u).$$

The parameters  $\lambda > \mu > 0$  denote the relaxation and retardation time, respectively, and  $\eta$  is the viscosity of the fluid.

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The tangential part of the stress tensor  $\tau$  can be decomposed as  $\tau = \tau_N + \tau_e$  where  $\tau_N = 2\eta\frac{\mu}{\lambda}D(u)$  corresponds to the Newtonian part and  $\tau_e$  to the purely elastic part. Denoting  $\tau_e = \tau$ , the above equations can be rewritten as

$$(1.4) \quad \begin{cases} \partial_t u + (u \cdot \nabla)u - \eta(1 - \omega)\Delta u + \nabla p &= \operatorname{div} \tau, \\ \operatorname{div} u &= 0, \\ \lambda(\partial_t \tau + (u \cdot \nabla)\tau + g_\alpha(\tau, \nabla u)) + \tau &= 2\eta\omega D(u), \end{cases}$$

with the coupling parameter  $\omega := 1 - \frac{\mu}{\lambda} \in (0, 1)$ . Using dimensionless variables, Oldroyd-B model can be thus described by the system (1.1). For more details of the modeling, c.f., [31, 12] for example.

The theory of Oldroyd-B fluids recently gained quite some attention. According to the coupling strength between the velocity  $u$  and the symmetric tensor of the constrains  $\tau$ , the results on Oldroyd-B model fall into two categories.

**(I) Weak coupling case: small coupling parameter  $\omega$ .** The study of the incompressible Oldroyd-B model was started by a pioneering paper given by Guillopé and Saut [15]. They proved that (i) system (1.1) admits a unique local strong solution in suitable Sobolev spaces  $H^s(\Omega)$  for *bounded domains*  $\Omega \subset \mathbb{R}^3$ ; and (ii) this solution is global provided the data as well as the coupling parameter  $\omega$  are sufficiently small. For extensions to this results to the  $L^p$ -setting, see the work of Fernandéz-Cara, Guillén and Ortega [13]. The situation of *exterior domains* was considered first by Hieber, Naito and Shibata [16], where the existence of a unique global strong solution defined in certain function spaces was proved provided the initial data and the coupling parameter  $\omega$  are small enough. Another remarkable way to study the Oldroyd-B model (1.1) is constructing solutions in *scaling invariant* spaces. This approach goes back to the seminal work by Fujita and Kato [14] for the classical incompressible Navier-Stokes equations. While for the Oldroyd-B model, strictly speaking, the system (1.1) does not have any scaling invariance. Fortunately, if we neglect the coupling term  $\operatorname{div} \tau$  and the damping term  $\tau$ , it is found that (1.1) is invariant under the transformation

$$(1.5) \quad \begin{cases} (u_0, \tau_0) \rightarrow (\ell u_0(\ell x), \tau_0(\ell x)), \\ (u(t, x), \tau(t, x), \Pi(t, x)) \rightarrow (\ell u(\ell x, \ell^2 t), \tau(\ell x, \ell^2 t), \ell^2 \Pi(\ell x, \ell^2 t)), \end{cases}$$

for any  $\ell > 0$ . Chemin and Masmodi [7] first study the Oldroyd-B model (1.1) along this line. For small coupling parameter  $\omega$ , they showed the global well-posedness of system (1.1) with initial data

$(u_0, \tau_0)$  belonging to the critical space  $\left(\dot{B}_{p,1}^{\frac{d}{p}-1}\right)^d \times \left(\dot{B}_{p,1}^{\frac{d}{p}}\right)^{d \times d}$  for any  $p \in [1, \infty)$ .

**(II) Strong coupling case: arbitrary coupling parameter  $\omega \in (0, 1)$ .** Oldroyd-B model with strong coupling parameter describes the behavior of the viscoelastic fluids more appropriately. The first result on the system (1.1) with strong coupling parameter was obtained by Molinet and Talhouk [27] for *bounded domains*. They removed the smallness restriction on the coupling parameter  $\omega$  in [15]. For the situation of *exterior domains*, Hieber and the authors recently [11] improved the main result given in [16] to the situation of non-small coupling parameter. Constructing solutions in *scaling invariant* spaces with *strong* coupling is more difficult. Some efforts were made by Chen and Miao in [8], where they established global solutions to the incompressible Oldroyd-B model with small initial data in  $B_{2,\infty}^s$ ,  $s > \frac{d}{2}$ . Later on, this result was generalized by the second author [34] with data in  $\dot{H}^s \cap \dot{B}_{2,1}^{\frac{d}{2}}, -\frac{d}{2} < s < \frac{d}{2} - 1$ . It was not until very recently that Zhang and the authors [33] obtained the scaling invariant solution with strong coupling in the critical  $L^p$  framework. For the *weak solutions* of incompressible Oldroyd-B fluids, see the work of Lions and Masmoudi [26] for the case  $\alpha = 0$ . The general case  $\alpha \neq 0$  is still open up to now. As for the *blow-up criteria* of the incompressible Oldroyd-B model, there are works [7, 18, 22]. Besides, we would like to mention that Constantin and Kliegl [9] proved the global regularity of solutions in two dimensional case for the Oldroyd-B fluids with *diffusive stress*. An approach based on *Lagrangian particle dynamics* can be found in [19, 20, 21, 23, 24, 25, 30, 32].

On the other hand, for the classical Navier-Stokes equations, taking advantage of the algebraical structure of the incompressible condition  $\operatorname{div} u = 0$ , some recent progresses have been made such as [36, 29, 5, 6]. Following these ideas, Zhang [37] proved the existence and uniqueness of solutions to system related to the incompressible viscoelastic fluids with a class of large initial data.

Motivated by [36, 37], the aim of this paper is to investigate the global well-posedness of system (1.1) in critical Besov spaces with the vertical component of the initial velocity field being large. Basically it is the algebraical structure of system (1.1) that enables us to consider this type of initial data. More precisely, using the divergence free condition  $\operatorname{div} u = 0$ , the vertical component of the convection term  $u \cdot \nabla u^d$  can be rewritten as  $u^h \partial_h u^d - u^d \operatorname{div}_h u^h$ . As a result, the vertical component  $u^d$  of the velocity satisfies a linear equation with coefficients depending on the horizontal components  $u^h$  and  $\tau$ . This is the reason why we do not believe we have to impose the smallness condition on the initial vertical velocity  $u_0^d$ . While the equation for the horizontal velocity  $u^h$  contains bilinear terms in the horizontal components and also terms describing the interactions between the horizontal components and the vertical one. This necessitates the smallness restriction on the initial horizontal velocity  $u_0^h$ . Besides, since the equation of the symmetric tensor of constrains  $\tau$  is nonlinear, considering the interactions between the velocity  $u$  (with large vertical component) and  $\tau$ , imposing a smallness condition on  $\tau_0$  is necessary. In addition, we would like to emphasize that both the equation of vertical velocity and horizontal velocity are linearly coupled with  $\tau$ , and conversely, the equation of the symmetric tensor of constrains  $\tau$  is linearly coupled with  $u$  as well. In particular, the coupling parameter  $\omega$  is involved. In order to decouple  $u$  and  $\tau$  linearly, one way to do is simply to apply the estimates for *heat equation* and *transport equation with damping* to the velocity  $u$  and the symmetric tensor of constrains  $\tau$  respectively, and to substitute the estimate for  $\tau$  to that for  $u$ . Nevertheless, in doing this, the coupling parameter  $\omega$  must be small enough to close the estimate of  $u$ . This less interesting *weak* coupling case is exactly what we want to avoid considering.

Although the *strong* coupling case with initial data in  $L^p$  type critical Besov space was investigated by us in [33], large initial vertical velocity is not allowed in [33]. A natural question is that can we solve system (1.1) with *strong* coupling parameter  $\omega$  and *large* initial vertical velocity  $u_0^d$ ? Before answering it, let us give some heuristic observations. Given large initial vertical velocity, it has been pointed out above that the horizontal velocity  $u^h$  and the symmetric tensor of constrains  $\tau$  should be small. In the circumstances, the linear term  $2\omega D(u)$  should also be small otherwise the equation  $(1.1)_2$  of  $\tau$  can not hold. Since we can not expect the vertical velocity to be small, it is reasonable to construction solutions to system (1.1) with the product  $\omega u^d$  of the coupling parameter  $\omega$  and the vertical velocity  $u^d$  small. That's why we need the initial product  $\omega u_0^d$  to be small in this paper.

Let us now explain the main ingredients of the proof. Noting that the algebraical structure of system (1.1) plays a key role in our result, the method used in [33] which relied heavily on the Green matrix of the linearized system of (1.1) and was carried out by Lagrangian approach fails to work here. It is because the divergence free condition does not hold anymore in Lagrangian coordinate. In this paper, we exploit energy approach in the critical  $L^2$  framework. As the damping  $\tau$  in the equation  $(1.1)_2$  may mislead us to the *weak* coupling case, so a crucial step would be to find a structural mechanism which indicates a parabolic smoothing effect on  $u$  in the absence of the damping effect on  $\tau$ . To the best of our knowledge, in  $L^2$  framework, researchers [27, 11, 8, 34] were inclined to use the cancelation relation

$$(1.6) \quad (\operatorname{div} \tau | u) + (D(u) | \tau) = 0$$

to eliminate the effect of the coupling parameter  $\omega$ . The price they had to pay was that the solution space served for  $u$  and  $\tau$  would be of the same regularity, which violated the scaling. So our strategy is not to estimate  $(u, \tau)$  as a whole with an eye to (1.6), but to find a new quantity which obeys the scaling and can be used as a substitute for  $\tau$ . An ideal candidate is  $\mathbb{P} \operatorname{div} \tau$  because on the one hand,  $\mathbb{P} \operatorname{div} \tau$  is the exact quantity linearly coupled with  $u$  and obeying the scaling. On the other hand, it has

been shown in [33] that the Green matrix  $G(t, x)$  of the following auxiliary system of  $(u, \mathbb{P}\text{div}\tau)$

$$(1.7) \quad \begin{cases} \text{Re } u_t - (1 - \omega)\Delta u - \mathbb{P}\text{div}\tau &= -\text{Re } \mathbb{P}(u \cdot \nabla)u, \\ \text{We } \mathbb{P}\text{div}\tau_t + \mathbb{P}\text{div}\tau - \omega\Delta u &= -\text{We } (\mathbb{P}\text{div}(u \cdot \nabla)\tau + \mathbb{P}\text{div}g_\alpha(\tau, \nabla u)), \\ u(0) &= u_0, \\ \mathbb{P}\text{div}\tau(0) &= \mathbb{P}\text{div}\tau_0. \end{cases}$$

behaves just like the heat kernel in the low frequency case. In this paper, we will pay much attention to the system (1.7) in the spirit of Danchin [10]. More precisely, we first localize (1.7) in frequencies according to the Littlewood-Paley decomposition. Then a variant energy method will be used to estimate each dyadic block  $(\dot{\Delta}_q u, \dot{\Delta}_q \mathbb{P}\text{div}\tau)$ . Owing to the fact that the linear operator associated with (1.7) can not be diagonalized in a basis independent of  $\xi$ , coercive estimates can not be achieved by standard energy argument. Since the parabolic-hyperbolic system (1.7) behaves differently in high and low frequencies, we shall consider the following quantity (for some  $q_0, q_1 \in \mathbb{Z}$  to be chosen later):

$$Y_q = \begin{cases} \left[ \|u_q\|_{L^2}^2 + 2 \left\| \frac{\text{We}}{\text{Re}} \mathbb{P}\text{div}\tau_q \right\|_{L^2}^2 + 2(u_q | \frac{\text{We}}{\text{Re}} \mathbb{P}\text{div}\tau_q) + 2(u_q | \frac{\text{We}}{\text{Re}} \Delta \mathbb{P}\text{div}\tau_q) \right]^{\frac{1}{2}}, & \text{for } q \leq q_1, \\ \left[ 2\|u_q\|_{L^2}^2 + \left\| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \mathbb{P}\text{div}\tau_q \right\|_{L^2}^2 - 2 \left( u_q | \frac{(1-\omega)\text{We}}{\omega\text{Re}} \mathbb{P}\text{div}\tau_q \right) \right]^{\frac{1}{2}}, & \text{for } q > q_0, \\ \left[ \|u_q\|_{L^2}^2 + \frac{\text{We}}{\omega\text{Re}} \|\Lambda^{-1} \mathbb{P}\text{div}\tau_q\|_{L^2}^2 \right]^{\frac{1}{2}}, & \text{for } q_1 < q \leq q_0. \end{cases}$$

Some words about the definition of  $Y_q$  are in order.

- For  $q \leq q_1$ , the parabolic smoothing effect of  $\mathbb{P}\text{div}\tau$  is obtained with the aid of the linear coupling term  $-\mathbb{P}\text{div}\tau$  in  $(1.7)_1$ , that's why we introduce the cross term  $(u_q | \frac{\text{We}}{\text{Re}} \Delta \mathbb{P}\text{div}\tau_q)$ .
- For  $q > q_0$ , compared with the linear coupling term  $-\mathbb{P}\text{div}\tau$  in  $(1.7)_1$ , the damping term  $\mathbb{P}\text{div}\tau$  in  $(1.7)_2$  is predominate, and thus there is a damping effect on  $\mathbb{P}\text{div}\tau$ .
- For  $q_1 < q \leq q_0$ , a new quantity  $\Lambda^{-1} \mathbb{P}\text{div}\tau$  is introduced to fill up the gap between  $q_1$  and  $q_0$ . Indeed, in this case, we do not need to care about the scaling thanks to Bernstein's inequalities. On this basis, the cancelation relation (1.6) motivates us to choose  $\Lambda^{-1} \mathbb{P}\text{div}\tau$  to replace  $\mathbb{P}\text{div}\tau$ . In doing so,  $u$  and  $\Lambda^{-1} \mathbb{P}\text{div}\tau$  are linearly decoupled, and  $\Lambda^{-1} \mathbb{P}\text{div}\tau$  also possesses a damping effect because of the damping term. Therefore, in this paper we combine the case  $q > q_0$  with  $q_1 < q \leq q_0$ , and regard them as high frequency. The left part  $q \leq q_1$  is regarded as low frequency.

We would like to remark that similar construction of  $Y_q$  was also used by the second author in [35] for the compressible Oldroyd-B model. However, for  $q_1 < q \leq q_0$ , the quantity in [35] corresponding to  $Y_q$  is

$$\tilde{Y}_q := \left[ \|\mathbb{P}u_q\|_{L^2}^2 + \frac{\text{We}}{\omega\text{Re}} \|\Lambda^{-1} \mathbb{P}\text{div}\tau_q\|_{L^2}^2 - \frac{2\beta(1-\omega)\text{We}}{\omega\text{Re}} (\mathbb{P}u_q | \mathbb{P}\text{div}\tau_q) \right]^{\frac{1}{2}},$$

with some positive constant  $\beta$ . Since the cross term  $(\mathbb{P}u_q | \mathbb{P}\text{div}\tau_q)$  is involved in  $\tilde{Y}_q$ ,  $\Lambda^{-1} \mathbb{P}\text{div}\tau$  in [35] exhibits the parabolic smoothing effect stemming from the linear coupling term  $-\mathbb{P}\text{div}\tau$ . Hence the case  $q_1 < q \leq q_0$  is regarded as low frequency in [35]. The point is, though, that  $\tilde{Y}_q$  can not be used in this paper to replace  $Y_q$  because the parabolic smoothing effect for  $\Lambda^{-1} \mathbb{P}\text{div}\tau$  and hence for  $u$  is too weak (depending on  $\omega$ ) to deal with the linear coupling term  $2\omega D(u)$  in  $(1.1)_2$ .

Obviously, the above analysis do not take into account the anisotropy. In order to distinguish the velocity  $u$  from different directions, we must bound  $(u^h, (\mathbb{P}\text{div}\tau)^h)$  and  $(u^d, (\mathbb{P}\text{div}\tau)^d)$  separately (please refer to Section 3 for more details).

We shall obtain the existence and uniqueness of a solution  $(u, \tau)$  to (1.1) in the following space.

**Definition 1.1.** For  $T > 0$ , and  $s \in \mathbb{R}$ , let us denote

$$\mathcal{E}_T^s := \left( \tilde{C}_T(\dot{B}_{2,1}^{s-1}) \cap L_T^1(\dot{B}_{2,1}^{s+1}) \right)^d \times \left( \tilde{C}_T(\dot{B}_{2,1}^s) \cap L_T^1(\dot{B}_{2,1}^s) \right)^{d \times d}.$$

We use the notation  $\mathcal{E}^s$  if  $T = \infty$ , changing  $[0, T]$  into  $[0, \infty)$  in the definition above.

Our main result is stated as follows:

**Theorem 1.1.** *Let  $d \geq 2$ . Assume that  $(u_0, \tau_0) \in \left(\dot{B}_{2,1}^{\frac{d-1}{2}}\right)^d \times \left(\dot{B}_{2,1}^{\frac{d}{2}}\right)^{d \times d}$  with  $\operatorname{div} u_0 = 0$ . There exists a positive constant  $M$ , depending on  $d, \operatorname{Re}$  and  $\operatorname{We}$ , but independent of  $\omega$ , such that if*

$$(1.8) \quad \frac{C_0}{(1-\omega)^9} \left( \|u_0^h\|_{\dot{B}_{2,1}^{\frac{d-1}{2}}} + \omega \|u_0^d\|_{\dot{B}_{2,1}^{\frac{d-1}{2}}} + \|\tau_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) \exp \left( \frac{C_0}{(1-\omega)^{12}} \|u_0^d\|_{\dot{B}_{2,1}^{\frac{d-1}{2}}}^2 \right) \leq 1,$$

then system (1.1) admits a unique global solution  $(u, \tau)$  in  $\mathcal{E}^{\frac{d}{2}}$  with

$$(1.9) \quad \begin{aligned} & \|u^d\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d-1}{2}})} + \|(\mathbb{P} \operatorname{div} \tau)^d\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d-1}{2}})} + \|u^d\|_{L_t^1(\dot{B}_{2,1}^{\frac{d+1}{2}})} + \|(\mathbb{P} \operatorname{div} \tau)^d\|_{L_t^1(\dot{B}_{2,1}^{\frac{d+1}{2}, \frac{d-1}{2}})} \\ & \leq \frac{C_0}{(1-\omega)^3} \left( 1 + \|u_0^d\|_{\dot{B}_{2,1}^{\frac{d-1}{2}}} + \|(\mathbb{P} \operatorname{div} \tau)_0^d\|_{\dot{B}_{2,1}^{\frac{d-1}{2}}} \right), \end{aligned}$$

and

$$(1.10) \quad \begin{aligned} & \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d-1}{2}})} + \|(\mathbb{P} \operatorname{div} \tau)^h\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d-1}{2}})} + \|u^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{d+1}{2}})} + \|(\mathbb{P} \operatorname{div} \tau)^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{d+1}{2}, \frac{d-1}{2}})} \\ & \leq \frac{C_0}{(1-\omega)^6} \left( \|u_0^h\|_{\dot{B}_{2,1}^{\frac{d-1}{2}}} + \omega \|u_0^d\|_{\dot{B}_{2,1}^{\frac{d-1}{2}}} + \|\tau_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) \exp \left( \frac{C_0}{(1-\omega)^{12}} \|u_0^d\|_{\dot{B}_{2,1}^{\frac{d-1}{2}}}^2 \right), \end{aligned}$$

and

$$(1.11) \quad \begin{aligned} & \|\tau\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} + \|\tau\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} \\ & \leq \frac{C_0}{(1-\omega)^6} \left( \|u_0^h\|_{\dot{B}_{2,1}^{\frac{d-1}{2}}} + \omega \|u_0^d\|_{\dot{B}_{2,1}^{\frac{d-1}{2}}} + \|\tau_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) \exp \left( \frac{C_0}{(1-\omega)^{12}} \|u_0^d\|_{\dot{B}_{2,1}^{\frac{d-1}{2}}}^2 \right). \end{aligned}$$

The rest part of this paper is organized as follows. In Section 2, we introduce the tools (the Littlewood-Paley decomposition and paradifferential calculus) and give some nonlinear estimates in Besov spaces. Section 3 is devoted to the global a priori estimates of system (1.1). The proof of Theorem 1.1 is given in Section 4.

### Notation.

- (1) For  $f \in \mathcal{S}'$ ,  $f_q := \dot{\Delta}_q f$ .
- (2) For  $x = (x_1, \dots, x_d)$ , we denote  $x_h := (x_1, \dots, x_{d-1})$ ; For vector field  $u(x) = (u^1(x), \dots, u^d(x))$ , we denote  $u^h(x) := (u^1(x), \dots, u^{d-1}(x))$ ,  $\partial_h u(x) := (\partial_1 u(x), \dots, \partial_{d-1} u(x))^\top$ , and  $\operatorname{div}_h u^h(x) := \partial_1 u^1(x) + \dots + \partial_{d-1} u^{d-1}(x)$ .
- (3) For  $a, b \in L^2$ ,  $(a|b)$  denotes the  $L^2$  inner product of  $a$  and  $b$ .
- (4) Throughout the paper,  $C$  denotes various ‘‘harmless’’ positive constants, and we sometimes use the notation  $A \lesssim B$  as an equivalent to  $A \leq CB$ . The notation  $A \approx B$  means that  $A \lesssim B$  and  $B \lesssim A$ .

## 2. THE FUNCTIONAL TOOL BOX

The results of the present paper rely on the use of a dyadic partition of unity with respect to the Fourier variables, the so-called the *Littlewood-Paley decomposition*. Let us briefly explain how it may be built in the case  $x \in \mathbb{R}^d$  which the readers may see more details in [1, 3]. Let  $(\chi, \varphi)$  be a couple of  $C^\infty$  functions satisfying

$$(2.1) \quad \operatorname{Supp} \chi \subset \{|\xi| \leq \frac{4}{3}\}, \quad \operatorname{Supp} \varphi \subset \{\frac{3}{4} \leq |\xi| \leq \frac{8}{3}\},$$

and

$$\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1,$$

$$\sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1, \quad \text{for } \xi \neq 0.$$

Set  $\varphi_q(\xi) = \varphi(2^{-q}\xi)$ ,  $h_q = \mathcal{F}^{-1}(\varphi_q)$ , and  $\tilde{h} = \mathcal{F}^{-1}(\chi)$ . The dyadic blocks and the low-frequency cutoff operators are defined for all  $q \in \mathbb{Z}$  by

$$\dot{\Delta}_q u = \varphi(2^{-q}\mathbf{D})u = \int_{\mathbb{R}^d} h_q(y)u(x-y)dy,$$

$$\dot{S}_q u = \chi(2^{-q}\mathbf{D})u = \int_{\mathbb{R}^d} \tilde{h}_q(y)u(x-y)dy.$$

Then

$$(2.2) \quad u = \sum_{q \in \mathbb{Z}} \dot{\Delta}_q u,$$

holds for tempered distributions *modulo polynomials*. As working modulo polynomials is not appropriate for nonlinear problems, we shall restrict our attention to the set  $\mathcal{S}'_h$  of tempered distributions  $u$  such that

$$\lim_{q \rightarrow -\infty} \|\dot{S}_q u\|_{L^\infty} = 0.$$

Note that (2.2) holds true whenever  $u$  is in  $\mathcal{S}'_h$  and that one may write

$$\dot{S}_q u = \sum_{p \leq q-1} \dot{\Delta}_p u.$$

Besides, we would like to mention that the Littlewood-Paley decomposition has a nice property of quasi-orthogonality:

$$(2.3) \quad \dot{\Delta}_p \dot{\Delta}_q u \equiv 0 \text{ if } |p-q| \geq 2 \text{ and } \dot{\Delta}_p(\dot{S}_{q-1} u \dot{\Delta}_q u) \equiv 0 \text{ if } |p-q| \geq 5.$$

One can now give the definition of homogeneous Besov spaces.

**Definition 2.1.** For  $s \in \mathbb{R}$ ,  $(p, r) \in [1, \infty]^2$ , and  $u \in \mathcal{S}'(\mathbb{R}^d)$ , we set

$$\|u\|_{\dot{B}_{p,r}^s} = \|2^{sq} \|\dot{\Delta}_q u\|_{L^p}\|_{\ell^r}.$$

We then define the space  $\dot{B}_{p,r}^s := \{u \in \mathcal{S}'_h(\mathbb{R}^d), \|u\|_{\dot{B}_{p,r}^s} < \infty\}$ .

Since homogeneous Besov spaces fail to have nice inclusion properties, it is wise to define *hybrid Besov spaces* where the growth conditions satisfied by the dyadic blocks are different for low and high frequencies. In fact, hybrid Besov spaces played a crucial role for proving global well-posedness of compressible barotropic Navier-Stokes equations in critical spaces [10]. Let us now define the hybrid Besov spaces that we need. Here our notations are somehow different from those in [10].

**Definition 2.2.** Let  $s, t \in \mathbb{R}$ , and  $u \in \mathcal{S}'(\mathbb{R}^d)$ . For some fixed  $q_0 \in \mathbb{Z}$ , we set

$$\|u\|_{\dot{B}_{2,1}^{s,t}} := \sum_{q \leq q_0} 2^{qs} \|\dot{\Delta}_q u\|_{L^2} + \sum_{q > q_0} 2^{qt} \|\dot{\Delta}_q u\|_{L^2}.$$

We then define the space  $\dot{B}_{2,1}^{s,t} := \{u \in \mathcal{S}'_h(\mathbb{R}^d), \|u\|_{\dot{B}_{2,1}^{s,t}} < \infty\}$ .

**Remark 2.1.** For all  $s, t \in \mathbb{R}$ ,  $\tilde{q}_0 \in \mathbb{Z}$ , and  $u \in \mathcal{S}'(\mathbb{R}^d)$ , setting

$$\|u\|_{\tilde{B}_{2,1}^{s,t}} := \sum_{q \leq \tilde{q}_0} 2^{qs} \|\dot{\Delta}_q u\|_{L^2} + \sum_{q > \tilde{q}_0} 2^{qt} \|\dot{\Delta}_q u\|_{L^2},$$

then it is easy to verify that  $\|u\|_{\tilde{B}_{2,1}^{s,t}} \approx \|u\|_{\dot{B}_{2,1}^{s,t}}$ .

The following lemma describes the way derivatives act on spectrally localized functions.

**Lemma 2.1** (Bernstein's inequalities). *Let  $k \in \mathbb{N}$  and  $0 < r < R$ . There exists a constant  $C$  depending on  $r, R$  and  $d$  such that for all  $(a, b) \in [1, \infty]^2$ , we have for all  $\lambda > 0$  and multi-index  $\alpha$*

- *If  $\text{Supp } \hat{f} \subset B(0, \lambda R)$ , then  $\sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^b} \leq C^{k+1} \lambda^{k+d(\frac{1}{a}-\frac{1}{b})} \|f\|_{L^a}$ .*
- *If  $\text{Supp } \hat{f} \subset C(0, \lambda r, \lambda R)$ , then  $C^{-k-1} \lambda^k \|f\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^a} \leq C^{k+1} \lambda^k \|f\|_{L^a}$ .*

In fact, the following special case of Lemma 2.1 will be used very frequently in this paper.

**Corollary 2.1.** *Let  $\Lambda := \sqrt{-\Delta}$ , then for any  $f \in \mathcal{S}'$ , there holds*

$$(2.4) \quad \frac{3}{4} 2^q \|\dot{\Delta}_q f\|_{L^2} \leq \|\Lambda \dot{\Delta}_q f\|_{L^2} \leq \frac{8}{3} 2^q \|\dot{\Delta}_q f\|_{L^2}.$$

Owing to the property (2.1) of  $\text{Supp } \varphi$ , (2.4) can be easily obtained by virtue of Plancherel's Theorem.

Let us now state some classical properties for the Besov spaces.

**Proposition 2.1.** *For all  $s, s_1, s_2 \in \mathbb{R}$ ,  $1 \leq p, p_1, p_2, r, r_1, r_2 \leq \infty$ , the following properties hold true:*

- *If  $p_1 \leq p_2$  and  $r_1 \leq r_2$ , then  $\dot{B}_{p_1, r_1}^s \hookrightarrow \dot{B}_{p_2, r_2}^{s-\frac{d}{p_1}+\frac{d}{p_2}}$ .*
- *If  $s_1 \neq s_2$  and  $\theta \in (0, 1)$ ,  $[\dot{B}_{p, r_1}^{s_1}, \dot{B}_{p, r_2}^{s_2}]_{(\theta, r)} = \dot{B}_{p, r}^{\theta s_1 + (1-\theta)s_2}$ .*
- *For any smooth homogeneous of degree  $m \in \mathbb{Z}$  function  $F$  on  $\mathbb{R}^d \setminus \{0\}$ , the operator  $F(D)$  maps  $\dot{B}_{p, r}^s$  in  $\dot{B}_{p, r}^{s-m}$ .*

Next we recall a few nonlinear estimates in Besov spaces which may be obtained by means of paradifferential calculus. Firstly introduced by J. M. Bony in [2], the paraproduct between  $f$  and  $g$  is defined by

$$\dot{T}fg = \sum_{q \in \mathbb{Z}} \dot{S}_{q-1} f \dot{\Delta}_q g,$$

and the remainder is given by

$$\dot{R}(f, g) = \sum_{q \geq -1} \dot{\Delta}_q f \tilde{\Delta}_q g$$

with

$$\tilde{\Delta}_q g = (\dot{\Delta}_{q-1} + \dot{\Delta}_q + \dot{\Delta}_{q+1})g.$$

We have the following so-called Bony's decomposition:

$$(2.5) \quad fg = \dot{T}fg + \dot{T}_g f + \dot{R}(f, g) = \dot{T}fg + \dot{T}'_g f,$$

where  $\dot{T}'_g f := \dot{T}_g f + \dot{R}(f, g)$ . The paraproduct  $\dot{T}$  and the remainder  $\dot{R}$  operators satisfy the following continuous properties.

**Proposition 2.2.** *For all  $s \in \mathbb{R}$ ,  $\sigma > 0$ , and  $1 \leq p, p_1, p_2, r, r_1, r_2 \leq \infty$ , the paraproduct  $\dot{T}$  is a bilinear, continuous operator from  $L^\infty \times \dot{B}_{p, r}^s$  to  $\dot{B}_{p, r}^s$  and from  $\dot{B}_{\infty, r_1}^{s-\sigma} \times \dot{B}_{p, r_2}^s$  to  $\dot{B}_{p, r}^{s-\sigma}$  with  $\frac{1}{r} = \min\{1, \frac{1}{r_1} + \frac{1}{r_2}\}$ . The remainder  $\dot{R}$  is bilinear continuous from  $\dot{B}_{p_1, r_1}^{s_1} \times \dot{B}_{p_2, r_2}^{s_2}$  to  $\dot{B}_{p, r}^{s_1+s_2}$  with  $s_1 + s_2 > 0$ ,  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1$ , and  $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \leq 1$ .*

In view of (2.5), Proposition 2.2 and Bernstein's inequalities, one easily deduces the following product estimates:

**Corollary 2.2.** *Let  $p \in [1, \infty]$ . If  $s_1, s_2 \leq \frac{d}{p}$  and  $s_1 + s_2 > d \max\{0, \frac{2}{p} - 1\}$ , then there holds*

$$(2.6) \quad \|uv\|_{\dot{B}_{p, 1}^{s_1+s_2-\frac{d}{p}}} \leq C \|u\|_{\dot{B}_{p, 1}^{s_1}} \|v\|_{\dot{B}_{p, 1}^{s_2}}.$$

In the following, we shall give a commutator estimate, which will be used to deal with the convection terms.

**Lemma 2.2** ([35]). *Let  $\Lambda := \sqrt{-\Delta}$ , then*

$$(2.7) \quad \|[\Lambda^{-1}, \dot{S}_{q-1} v \cdot \nabla] \dot{\Delta}_q u\|_{L^2} \leq C \|\nabla \dot{S}_{q-1} v\|_{L^\infty} \|\Lambda^{-1} \dot{\Delta}_q u\|_{L^2}.$$

The divergence free condition allows us to get the following lemma, which plays a key role in this paper.

**Lemma 2.3** ([37]). *Let  $v$  be a divergence free vector field. We then have*

$$\|v^d\|_{L_{x_h}^2(L_{x_d}^\infty)}^2 \leq C \|\operatorname{div}_h v^h\|_{L^2} \|v^d\|_{L^2}, \quad \text{and} \quad \|v^d\|_{L^\infty} \leq C \|v^h\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{\frac{1}{2}} \|v^d\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{\frac{1}{2}}.$$

The study of non-stationary PDEs requires spaces of the type  $L_T^\rho(X) = L^\rho(0, T; X)$  for appropriate Banach spaces  $X$ . In our case, we expect  $X$  to be a Besov space, so that it is natural to localize the equations through Littlewood-Paley decomposition. We then get estimates for each dyadic block and perform integration in time. But, in doing so, we obtain the bounds in spaces which are not of the type  $L^\rho(0, T; \dot{B}_{p,r}^s)$ . That naturally leads to the following definition introduced by Chemin and Lerner in [4].

**Definition 2.3.** *For  $\rho \in [1, +\infty]$ ,  $s \in \mathbb{R}$ , and  $T \in (0, +\infty)$ , we set*

$$\|u\|_{\tilde{L}_T^\rho(\dot{B}_{p,r}^s)} = \left\| 2^{qs} \|\dot{\Delta}_q u(t)\|_{L_T^\rho(L^p)} \right\|_{\ell^r}$$

and denote by  $\tilde{L}_T^\rho(\dot{B}_{p,r}^s)$  the subset of distributions  $u \in \mathcal{S}'([0, T] \times \mathbb{R}^N)$  with finite  $\|u\|_{\tilde{L}_T^\rho(\dot{B}_{p,r}^s)}$  norm. When  $T = +\infty$ , the index  $T$  is omitted. We further denote  $\tilde{C}_T(\dot{B}_{p,r}^s) = C([0, T]; \dot{B}_{p,r}^s) \cap \tilde{L}_T^\infty(\dot{B}_{p,r}^s)$ .

**Remark 2.2.** *All the properties of continuity for the paraproduct, remainder, and product remain true for the Chemin-Lerner spaces. The exponent  $\rho$  just has to behave according to Hölder's inequality for the time variable.*

**Remark 2.3.** *The spaces  $\tilde{L}_T^\rho(\dot{B}_{p,r}^s)$  can be linked with the classical space  $L_T^\rho(\dot{B}_{p,r}^s)$  via the Minkowski inequality:*

$$\|u\|_{\tilde{L}_T^\rho(\dot{B}_{p,r}^s)} \leq \|u\|_{L_T^\rho(\dot{B}_{p,r}^s)} \quad \text{if } r \geq \rho, \quad \|u\|_{\tilde{L}_T^\rho(\dot{B}_{p,r}^s)} \geq \|u\|_{L_T^\rho(\dot{B}_{p,r}^s)} \quad \text{if } r \leq \rho.$$

### 3. A PRIORI ESTIMATES

We begin this section by rewriting the system (1.1) in term of  $(u^i, (\mathbb{P} \operatorname{div} \tau)^i)$ ,  $1 \leq i \leq d$ , as follows

$$(3.1) \quad \begin{cases} \partial_t u^i - \frac{1-\omega}{\operatorname{Re}} \Delta u^i - \frac{1}{\operatorname{Re}} (\mathbb{P} \operatorname{div} \tau)^i = f^i, \\ \partial_t (\mathbb{P} \operatorname{div} \tau)^i + \frac{1}{\operatorname{We}} (\mathbb{P} \operatorname{div} \tau)^i - \frac{\omega}{\operatorname{We}} \Delta u^i = h^i, \end{cases}$$

where  $f^i := -(\mathbb{P}(u \cdot \nabla)u)^i$ , and  $h^i := -(\mathbb{P} \operatorname{div}(u \cdot \nabla) \tau + \mathbb{P} \operatorname{div}_a(\tau, \nabla u))^i$ . For simplicity, let us denote  $\sigma := \mathbb{P} \operatorname{div} \tau$ , then system (3.1) can be localized as

$$(3.2) \quad \begin{cases} \partial_t u_q^i + \dot{S}_{q-1} u \cdot \nabla u_q^i - \frac{1-\omega}{\operatorname{Re}} \Delta u_q^i - \frac{1}{\operatorname{Re}} \sigma_q^i = \tilde{f}_q^i, \\ \partial_t \sigma_q^i + \dot{S}_{q-1} u \cdot \nabla \sigma_q^i + \frac{1}{\operatorname{We}} \sigma_q^i - \frac{\omega}{\operatorname{We}} \Delta u_q^i = \tilde{h}_q^i, \end{cases}$$

where  $\tilde{f}_q^i := \dot{S}_{q-1} u \cdot \nabla u_q^i + f_q^i$ , and  $\tilde{h}_q^i := \dot{S}_{q-1} u \cdot \nabla \sigma_q^i + h_q^i$ . Denote

$$\begin{aligned} A^h(t) &:= \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|\sigma^h\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|u^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} + \|\sigma^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1, \frac{d}{2}-1})}, \\ A^d(t) &:= \|u^d\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|\sigma^d\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|u^d\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} + \|\sigma^d\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1, \frac{d}{2}-1})}, \\ A^h(0) &:= \|u_0^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|\sigma_0^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}, \quad A^d(0) := \|u_0^d\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|\sigma_0^d\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}, \\ B(t) &:= \|\tau\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} + \|\tau\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}, \quad B(0) := \|\tau_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}}. \end{aligned}$$

We will establish the following proposition.



**Proposition 3.1.** *Let  $(u, \tau)$  be a solution to system (1.1). There exist constants  $C_1, C_2$  and  $C_3$  depending on  $d, \text{Re}$ , and  $\text{We}$  but independent of  $\omega$ , such that the following inequalities hold*

$$(3.3) \quad \begin{aligned} A^h(t) \leq & \frac{C_1}{(1-\omega)^3} \left( A^h(0) + \int_0^t \left( \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right) \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}}} dt' \right. \\ & + \int_0^t \left( \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^{\frac{1}{2}} \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^{\frac{1}{2}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \right) dt' \\ & \left. + \int_0^t \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} dt' \right), \end{aligned}$$

$$(3.4) \quad \begin{aligned} A^d(t) \leq & \frac{C_2}{(1-\omega)^3} \left( A^d(0) + \int_0^t \left( \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right) \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}}} dt' \right. \\ & + \int_0^t \left( \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^{\frac{1}{2}} \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^{\frac{1}{2}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \right) dt' \\ & \left. + \int_0^t \left( \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \right) \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} dt' \right), \end{aligned}$$

and

$$(3.5) \quad B(t) \leq C_3 \left( B(0) + \omega \left( \|u^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} + \|u^d\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \right) + \int_0^t (\|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}) \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}}} dt' \right).$$

*Proof.* For  $1 \leq i \leq d$ , let us denote

$$Y_q^i := \left[ \|u_q^i\|_{L^2}^2 + 2 \left\| \frac{\text{We}}{\text{Re}} \sigma_q^i \right\|_{L^2}^2 + 2(u_q^i | \frac{\text{We}}{\text{Re}} \sigma_q^i) + 2(u_q^i | \frac{\text{We}}{\text{Re}} \Delta \sigma_q^i) \right]^{\frac{1}{2}}, \quad \text{for } q \leq q_1,$$

$$Y_q^i := \left[ 2\|u_q^i\|_{L^2}^2 + \left\| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right\|_{L^2}^2 - 2 \left( u_q^i | \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right) \right]^{\frac{1}{2}}, \quad \text{for } q > q_0,$$

$$Y_q^i := \left[ \|u_q^i\|_{L^2}^2 + \frac{\text{We}}{\omega\text{Re}} \|\Lambda^{-1} \sigma_q^i\|_{L^2}^2 \right]^{\frac{1}{2}}, \quad \text{for } q_1 < q \leq q_0,$$

where  $q_0$  and  $q_1$  are to be determined later.

**Estimates of  $Y_q^i$ ,  $q \in \mathbb{Z}$ .**

*Step 1: low frequencies:  $q \leq q_1$ .*

Taking the  $L^2$  inner product of (3.2)<sub>1</sub> and (3.2)<sub>2</sub> with  $u_q^i$  and  $\sigma_q^i$  respectively, using the divergence free condition  $\text{div} u = 0$ , we obtain

$$(3.6) \quad \frac{1}{2} \frac{d}{dt} \|u_q^i\|_{L^2}^2 + \frac{1-\omega}{\text{Re}} \|\Lambda u_q^i\|_{L^2}^2 - \frac{1}{\text{Re}} (u_q^i | \sigma_q^i) = (\tilde{f}_q^i | u_q^i),$$

and

$$(3.7) \quad \frac{1}{2} \frac{d}{dt} \|\sigma_q^i\|_{L^2}^2 + \frac{1}{\text{We}} \|\sigma_q^i\|_{L^2}^2 - \frac{\omega}{\text{We}} (\Delta u_q^i | \sigma_q^i) = (\tilde{h}_q^i | \sigma_q^i).$$

To cancel the “bad” cross term  $(u_q^i | \sigma_q^i)$  in (3.6), we need the following equality for the  $L^2$  inner product  $(u_q^i | \sigma_q^i)$  of  $u_q^i$  and  $\sigma_q^i$ ,

$$(3.8) \quad \frac{d}{dt} (u_q^i | \sigma_q^i) + \frac{\omega}{\text{We}} \|\Lambda u_q^i\|_{L^2}^2 - \frac{1}{\text{Re}} \|\sigma_q^i\|_{L^2}^2 - \frac{1-\omega}{\text{Re}} (\Delta u_q^i | \sigma_q^i) + \frac{1}{\text{We}} (u_q^i | \sigma_q^i) = (\tilde{f}_q^i | \sigma_q^i) + (\tilde{h}_q^i | u_q^i),$$

which is nothing but the sum of  $((3.2)_1|\sigma_q^i)$  and  $((3.2)_2|u_q^i)$ . Multiplying (3.8) by  $\frac{We}{Re}$ , adding the resulting equality to (3.6), one deduces that

$$(3.9) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \|u_q^i\|_{L^2}^2 + 2(u_q^i | \frac{We}{Re} \sigma_q^i) \right) + \frac{1}{Re} \|\Lambda u_q^i\|_{L^2}^2 - \frac{We}{(Re)^2} \|\sigma_q^i\|_{L^2}^2 \\ & - \frac{1-\omega}{Re} (\Delta u_q^i | \frac{We}{Re} \sigma_q^i) = (\tilde{f}_q^i | u_q^i) + \frac{We}{Re} ((\tilde{f}_q^i | \sigma_q^i) + (\tilde{h}_q^i | u_q^i)). \end{aligned}$$

Multiplying (3.7) by  $2(\frac{We}{Re})^2$ , adding the resulting equality to (3.9), then the negative term  $-\frac{We}{(Re)^2} \|\sigma_q^i\|_{L^2}^2$  in (3.9) is absorbed, and we have

$$(3.10) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[ \|u_q^i\|_{L^2}^2 + 2\|\frac{We}{Re} \sigma_q^i\|_{L^2}^2 + 2(u_q^i | \frac{We}{Re} \sigma_q^i) \right] \\ & + \frac{1}{Re} \|\Lambda u_q^i\|_{L^2}^2 + \frac{1}{We} \|\frac{We}{Re} \sigma_q^i\|_{L^2}^2 - \frac{1+\omega}{Re} (\Delta u_q^i | \frac{We}{Re} \sigma_q^i) \\ & = (\tilde{f}_q^i | u_q^i) + 2\left(\frac{We}{Re}\right)^2 (\tilde{h}_q^i | \sigma_q^i) + \frac{We}{Re} ((\tilde{f}_q^i | \sigma_q^i) + (\tilde{h}_q^i | u_q^i)), \end{aligned}$$

In order to exhibit the smoothing effect of  $\sigma^i$ ,  $1 \leq i \leq d$  in the low frequency case, we shall make use of the linear coupling term  $-\frac{1}{Re} \sigma^i$  in the first equation of (3.1). To do this, taking the  $L^2$  inner product of  $(3.2)_1$  with  $\Delta \sigma_q^i$ , and taking the  $L^2$  inner product of  $(3.2)_2$  with  $\Delta u_q^i$ , then adding them together, we are led to

$$(3.11) \quad \begin{aligned} & \frac{d}{dt} (u_q^i | \Delta \sigma_q^i) + \frac{1}{Re} \|\Delta \sigma_q^i\|_{L^2}^2 - \frac{\omega}{We} \|\Delta u_q^i\|_{L^2}^2 - \frac{1-\omega}{Re} (\Delta u_q^i | \Delta \sigma_q^i) + \frac{1}{We} (\Delta u_q^i | \sigma_q^i) \\ & = (\tilde{f}_q^i | \Delta \sigma_q^i) + (\tilde{h}_q^i | \Delta u_q^i) - (\dot{S}_{q-1} u \cdot \nabla u_q^i | \Delta \sigma_q^i) - (\dot{S}_{q-1} u \cdot \nabla \sigma_q^i | \Delta u_q^i). \end{aligned}$$

Multiplying (3.11) by  $\frac{We}{Re}$ , adding the resulting equality to (3.10) yields

$$(3.12) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[ \|u_q^i\|_{L^2}^2 + 2\|\frac{We}{Re} \sigma_q^i\|_{L^2}^2 + 2(u_q^i | \frac{We}{Re} \sigma_q^i) + 2(u_q^i | \frac{We}{Re} \Delta \sigma_q^i) \right] \\ & + \frac{1}{Re} \|\Lambda u_q^i\|_{L^2}^2 - \frac{\omega}{Re} \|\Delta u_q^i\|_{L^2}^2 + \frac{1}{We} \|\frac{We}{Re} \sigma_q^i\|_{L^2}^2 + \frac{1}{We} \|\frac{We}{Re} \Delta \sigma_q^i\|_{L^2}^2 \\ & - \frac{1-\omega}{Re} (\Delta u_q^i | \frac{We}{Re} \Delta \sigma_q^i) + \left( \frac{1}{We} - \frac{1+\omega}{Re} \right) (\Delta u_q^i | \frac{We}{Re} \sigma_q^i) \\ & = (\tilde{f}_q^i | u_q^i) + 2\left(\frac{We}{Re}\right)^2 (\tilde{h}_q^i | \sigma_q^i) + \frac{We}{Re} ((\tilde{f}_q^i | \sigma_q^i) + (\tilde{h}_q^i | u_q^i)) \\ & + \frac{We}{Re} ((\tilde{f}_q^i | \Delta \sigma_q^i) + (\tilde{h}_q^i | \Delta u_q^i) - (\dot{S}_{q-1} u \cdot \nabla u_q^i | \Delta \sigma_q^i) - (\dot{S}_{q-1} u \cdot \nabla \sigma_q^i | \Delta u_q^i)), \end{aligned}$$

Let us now estimate the cross terms contained in  $Y_q^i$  for  $q \leq q_1$ . Indeed, using Cauchy-Schwarz inequality and (2.4), one easily deduces that

$$2|(u_q^i | \frac{We}{Re} \sigma_q^i)| \leq \frac{5}{8} \|u_q^i\|_{L^2}^2 + \frac{8}{5} \|\frac{We}{Re} \sigma_q^i\|_{L^2}^2,$$

and

$$2|(u_q^i | \frac{We}{Re} \Delta \sigma_q^i)| \leq \left( \frac{8}{3} 2^{q_1} \right)^2 \left( \|u_q^i\|_{L^2}^2 + \|\frac{We}{Re} \sigma_q^i\|_{L^2}^2 \right) \leq \frac{1}{4} \left( \|u_q^i\|_{L^2}^2 + \|\frac{We}{Re} \sigma_q^i\|_{L^2}^2 \right),$$

provided

$$(3.13) \quad 2^{q_1} \leq \frac{3}{16}.$$

Consequently,

$$\|u_q^i\|_{L^2}^2 + 2\|\frac{We}{Re} \sigma_q^i\|_{L^2}^2 + 2(u_q^i | \frac{We}{Re} \sigma_q^i) + 2(u_q^i | \frac{We}{Re} \Delta \sigma_q^i) \geq \frac{1}{8} \|u_q^i\|_{L^2}^2 + \frac{3}{20} \|\frac{We}{Re} \sigma_q^i\|_{L^2}^2,$$

and hence

$$(3.14) \quad (Y_q^i)^2 \approx \|u_q^i\|_{L^2}^2 + \left\| \frac{\text{We}}{\text{Re}} \sigma_q^i \right\|_{L^2}^2, \quad \text{if } q \leq q_1,$$

with  $q_1$  satisfying (3.13). Thanks to (2.4), the remaining two cross terms on the left hand side of (3.12) can be bounded as follows:

$$(3.15) \quad \begin{aligned} \left| \left( \frac{1}{\text{We}} - \frac{1+\omega}{\text{Re}} \right) (\Delta u_q^i | \frac{\text{We}}{\text{Re}} \sigma_q^i) \right| &\leq 2 \frac{\text{Re} + \text{We}}{\text{WeRe}} \frac{8}{3} 2^{q_1} \|\Delta u_q^i\|_{L^2} \left\| \frac{\text{We}}{\text{Re}} \sigma_q^i \right\|_{L^2} \\ &\leq \frac{8}{3} 2^{q_1} \frac{\text{Re} + \text{We}}{\sqrt{\text{WeRe}}} \left( \frac{1}{\text{Re}} \|\Delta u_q^i\|_{L^2}^2 + \frac{1}{\text{We}} \left\| \frac{\text{We}}{\text{Re}} \sigma_q^i \right\|_{L^2}^2 \right) \\ &\leq \frac{1}{4} \left( \frac{1}{\text{Re}} \|\Delta u_q^i\|_{L^2}^2 + \frac{1}{\text{We}} \left\| \frac{\text{We}}{\text{Re}} \sigma_q^i \right\|_{L^2}^2 \right), \end{aligned}$$

provided

$$(3.16) \quad 2^{q_1} \leq \frac{3}{32} \frac{\sqrt{\text{WeRe}}}{\text{Re} + \text{We}}.$$

Moreover,

$$(3.17) \quad \begin{aligned} \frac{1-\omega}{\text{Re}} \left| (\Delta u_q^i | \frac{\text{We}}{\text{Re}} \Delta \sigma_q^i) \right| &\leq \frac{1}{2} \left( \frac{8}{3} 2^{q_1} \right)^2 \frac{1}{\text{Re}} \left( \|\Delta u_q^i\|_{L^2}^2 + \left\| \frac{\text{We}}{\text{Re}} \Delta \sigma_q^i \right\|_{L^2}^2 \right) \\ &\leq \frac{1}{8} \frac{1}{\text{Re}} \|\Delta u_q^i\|_{L^2}^2 + \frac{1}{2} \frac{1}{\text{We}} \left\| \frac{\text{We}}{\text{Re}} \Delta \sigma_q^i \right\|_{L^2}^2, \end{aligned}$$

provided

$$(3.18) \quad 2^{q_1} \leq \frac{3}{16}, \quad \text{and} \quad 2^{q_1} \leq \frac{3}{8} \sqrt{\frac{\text{Re}}{\text{We}}}.$$

Finally, using (2.4) again yields

$$(3.19) \quad \frac{\omega}{\text{Re}} \|\Delta u_q^i\|_{L^2}^2 \leq \left( \frac{8}{3} 2^{q_1} \right)^2 \frac{1}{\text{Re}} \|\Delta u_q^i\|_{L^2}^2 \leq \frac{1}{4} \frac{1}{\text{Re}} \|\Delta u_q^i\|_{L^2}^2,$$

provided (3.13) holds. Now collecting (3.13), (3.16) and (3.18), we take

$$(3.20) \quad q_1 := \left\lceil \log_2 \left( \frac{3}{32} \frac{\sqrt{\text{WeRe}}}{\text{Re} + \text{We}} \right) \right\rceil.$$

From (3.15)–(3.19), and using the left hand side of (2.4), we find that, for  $q \leq q_1$  with  $q_1$  satisfying (3.20), there holds

$$(3.21) \quad \begin{aligned} &\frac{1}{\text{Re}} \|\Delta u_q^i\|_{L^2}^2 - \frac{\omega}{\text{Re}} \|\Delta u_q^i\|_{L^2}^2 + \frac{1}{\text{We}} \left\| \frac{\text{We}}{\text{Re}} \sigma_q^i \right\|_{L^2}^2 + \frac{1}{\text{We}} \left\| \frac{\text{We}}{\text{Re}} \Delta \sigma_q^i \right\|_{L^2}^2 \\ &- \frac{1-\omega}{\text{Re}} (\Delta u_q^i | \frac{\text{We}}{\text{Re}} \Delta \sigma_q^i) + \left( \frac{1}{\text{We}} - \frac{1+\omega}{\text{Re}} \right) (\Delta u_q^i | \frac{\text{We}}{\text{Re}} \sigma_q^i) \\ &\geq \frac{3}{8} \frac{1}{\text{Re}} \|\Delta u_q^i\|_{L^2}^2 + \frac{1}{2} \frac{1}{\text{We}} \left\| \frac{\text{We}}{\text{Re}} \Delta \sigma_q^i \right\|_{L^2}^2 + \frac{3}{4} \frac{1}{\text{We}} \left\| \frac{\text{We}}{\text{Re}} \sigma_q^i \right\|_{L^2}^2 \\ &\geq \frac{3}{8} \left( \frac{3}{4} 2^q \right)^2 \min \left\{ \frac{1}{\text{We}}, \frac{1}{\text{Re}} \right\} \left( \|\Delta u_q^i\|_{L^2}^2 + \left\| \frac{\text{We}}{\text{Re}} \sigma_q^i \right\|_{L^2}^2 \right). \end{aligned}$$

To bound the right hand side of (3.12), firstly integrating by parts and using the divergence free condition  $\text{div} u = 0$  yields

$$(3.22) \quad -(\dot{S}_{q-1} u \cdot \nabla u_q^i | \Delta \sigma_q^i) - (\dot{S}_{q-1} \cdot \nabla \sigma_q^i | \Delta u_q^i) = (\dot{S}_{q-1} \partial_j u^k | \partial_k u_q^i \partial_j \sigma_q^i + \partial_k \sigma_q^i \partial_j u_q^i).$$

Accordingly, using Bernstein's inequality, (3.20) and Lemma 2.3, we are led to

$$\begin{aligned} & \frac{\text{We}}{\text{Re}} \left| (\dot{S}_{q-1} u \cdot \nabla u_q^i | \Delta \sigma_q^i) + (\dot{S}_{q-1} \cdot \nabla \sigma_q^i | \Delta u_q^i) \right| \\ & \leq C 2^{2q_1} \|\nabla u\|_{L^\infty} \|u_q^i\|_{L^2} \left\| \frac{\text{We}}{\text{Re}} \sigma_q^i \right\|_{L^2} \leq C \|u^h\|^{\frac{1}{2}}_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|u\|^{\frac{1}{2}}_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \left( \|u_q^i\|_{L^2}^2 + \left\| \frac{\text{We}}{\text{Re}} \sigma_q^i \right\|_{L^2}^2 \right), \end{aligned}$$

which, together with (3.14), implies that

$$(3.23) \quad \text{R.H.S. of (3.12)} \leq C \left( \|\tilde{f}_q^i\|_{L^2} + \left\| \frac{\text{We}}{\text{Re}} \tilde{h}_q^i \right\|_{L^2} + \|u^h\|^{\frac{1}{2}}_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|u\|^{\frac{1}{2}}_{\dot{B}_{2,1}^{\frac{d}{2}+1}} Y_q^i \right) Y_q^i.$$

Substituting (3.23) into (3.12), we infer from (3.14) and (3.21) that there exists a constant  $C$  depending on  $d, \text{Re}, \text{We}$ , but independent of  $\omega$ , such that if  $q \leq q_1$ , there holds

$$(3.24) \quad \frac{d}{dt} (Y_q^i)^2 + 2^{2q} (Y_q^i)^2 \leq C Y_q^i \left( \|\tilde{f}_q^i\|_{L^2} + \|\tilde{h}_q^i\|_{L^2} + \|u^h\|^{\frac{1}{2}}_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|u\|^{\frac{1}{2}}_{\dot{B}_{2,1}^{\frac{d}{2}+1}} Y_q^i \right).$$

*Step 2: high frequencies:  $q > q_1$ .*

Part (i).  $q > q_0$ .

First of all, we would like to point out that, in the high frequency case, the cross term  $-\frac{\omega}{\text{We}} (\Delta u_q^i | \sigma_q^i)$  in (3.7) is “bad”. We will cancel it with the aid of (3.8) once more. To this end, multiplying (3.7) and (3.8) by  $\left(\frac{(1-\omega)\text{We}}{\omega\text{Re}}\right)^2$  and  $-\frac{(1-\omega)\text{We}}{\omega\text{Re}}$  respectively, then adding the resulting equalities together yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \left\| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right\|_{L^2}^2 - 2 \left( u_q^i \left| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right| \right) \right) - \frac{1-\omega}{\text{Re}} \|\Delta u_q^i\|_{L^2}^2 \\ & + \frac{1}{(1-\omega)\text{We}} \left\| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right\|_{L^2}^2 - \frac{1}{\text{We}} \left( u_q^i \left| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right| \right) \\ (3.25) \quad & = \left( \frac{(1-\omega)\text{We}}{\omega\text{Re}} \right)^2 (\tilde{h}_q^i | \sigma_q^i) - \frac{(1-\omega)\text{We}}{\omega\text{Re}} ((\tilde{f}_q^i | \sigma_q^i) + (\tilde{h}_q^i | u_q^i)). \end{aligned}$$

To absorb the negative term  $-\frac{1-\omega}{\text{Re}} \|\Delta u_q^i\|_{L^2}^2$  in (3.25), we multiply (3.6) by 2, and add the resulting equality to (3.25) to get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( 2 \|u_q^i\|_{L^2}^2 + \left\| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right\|_{L^2}^2 - 2 \left( u_q^i \left| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right| \right) \right) + \frac{1-\omega}{\text{Re}} \|\Delta u_q^i\|_{L^2}^2 \\ & + \frac{1}{(1-\omega)\text{We}} \left\| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right\|_{L^2}^2 - \frac{2}{\text{Re}} (\sigma_q^i | u_q^i) - \frac{1}{\text{We}} \left( u_q^i \left| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right| \right) \\ (3.26) \quad & = 2(\tilde{f}_q^i | u_q^i) + \left( \frac{(1-\omega)\text{We}}{\omega\text{Re}} \right)^2 (\tilde{h}_q^i | \sigma_q^i) - \frac{(1-\omega)\text{We}}{\omega\text{Re}} ((\tilde{f}_q^i | \sigma_q^i) + (\tilde{h}_q^i | u_q^i)). \end{aligned}$$

It is easy to verify that

$$(3.27) \quad 2 \|u_q^i\|_{L^2}^2 + \left\| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right\|_{L^2}^2 - 2 \left( u_q^i \left| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right| \right) \approx \|u_q^i\|_{L^2}^2 + \left\| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right\|_{L^2}^2.$$

Using Cauchy-Schwarz inequality and (2.4) yields

$$\begin{aligned} & \left| \frac{2}{\text{Re}} (\sigma_q^i | u_q^i) + \frac{1}{\text{We}} \left( u_q^i \left| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right| \right) \right| \\ & = \frac{1+\omega}{1-\omega} \frac{1}{\text{We}} \left| \left( u_q^i \left| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right| \right) \right| \\ & \leq \frac{1}{2} \frac{1}{1-\omega} \frac{1}{\text{We}} \left\| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right\|_{L^2}^2 + \frac{2}{(1-\omega)\text{We}} \|u_q^i\|_{L^2}^2 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{2} \frac{1}{1-\omega} \frac{1}{\text{We}} \left\| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right\|_{L^2}^2 + \left( \frac{4}{3} 2^{-q_0} \right)^2 \frac{2}{(1-\omega)\text{We}} \|\Lambda u_q^i\|_{L^2}^2 \\
 (3.28) \quad &\leq \frac{1}{2} \frac{1}{1-\omega} \frac{1}{\text{We}} \left\| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right\|_{L^2}^2 + \frac{1}{2} \frac{1-\omega}{\text{Re}} \|\Lambda u_q^i\|_{L^2}^2,
 \end{aligned}$$

provided

$$(3.29) \quad \left( \frac{4}{3} 2^{-q_0} \right)^2 \frac{2}{(1-\omega)\text{We}} \leq \frac{1}{2} \frac{1-\omega}{\text{Re}}, \quad \text{i.e.} \quad 2^{q_0} \geq \frac{8}{3(1-\omega)} \sqrt{\frac{\text{Re}}{\text{We}}}.$$

Now let us take

$$(3.30) \quad q_0 := \left\lceil \log_2 \left( \frac{8}{3(1-\omega)} \sqrt{\frac{\text{Re}}{\text{We}}} \right) \right\rceil + 1.$$

Then it follows from (3.28), (3.29) and (2.4) that

$$\begin{aligned}
 &\frac{1-\omega}{\text{Re}} \|\Lambda u_q^i\|_{L^2}^2 + \frac{1}{(1-\omega)\text{We}} \left\| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right\|_{L^2}^2 \\
 &\quad - \frac{2}{\text{Re}} (\sigma_q^i | u_q^i) - \frac{1}{\text{We}} \left( u_q^i \left| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right. \right) \\
 (3.31) \quad &\geq \frac{1}{2} \frac{1-\omega}{\text{Re}} \|\Lambda u_q^i\|_{L^2}^2 + \frac{1}{2} \frac{1}{(1-\omega)\text{We}} \left\| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right\|_{L^2}^2 \\
 &\geq \frac{1}{2} \frac{1-\omega}{\text{Re}} \left( \frac{3}{4} 2^{q_0} \right)^2 \|u_q^i\|_{L^2}^2 + \frac{1}{2} \frac{1}{(1-\omega)\text{We}} \left\| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right\|_{L^2}^2 \\
 &\geq \frac{1}{2} \frac{1}{(1-\omega)\text{We}} \left( \|u_q^i\|_{L^2}^2 + \left\| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right\|_{L^2}^2 \right).
 \end{aligned}$$

Finally, the right hand side of (3.26) can be bounded as follows,

$$(3.32) \quad \text{R.H.S. of (3.26)} \leq C \left( \|\tilde{f}_q^i\|_{L^2} + \left\| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \tilde{h}_q^i \right\|_{L^2} \right) \left( \|u_q^i\|_{L^2} + \left\| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \sigma_q^i \right\|_{L^2} \right).$$

Substituting (3.32) into (3.26), using (3.27) and (3.31), we find that there exists a constant  $C$  independent of  $d, \text{Re}, \text{We}$ , and  $\omega$ , such that if  $q > q_0$ , there holds

$$(3.33) \quad \frac{d}{dt} (Y_q^i)^2 + \frac{1}{(1-\omega)\text{We}} (Y_q^i)^2 \leq C Y_q^i \left( \|\tilde{f}_q^i\|_{L^2} + \left\| \frac{(1-\omega)\text{We}}{\omega\text{Re}} \tilde{h}_q^i \right\|_{L^2} \right).$$

Part (ii).  $q_1 < q \leq q_0$ .

From the definitions of  $q_1$  and  $q_0$  (see (3.20) and (3.30)), it is obvious that there is a gap between the low frequency part  $q \leq q_1$  and the high frequency part  $q > q_0$  since  $q_1 < q_0$ . To overcome this difficulty, it is necessary to resort to some new observation. Considering that  $\|\Lambda^{-1} \sigma_q^i\|_{L^2}$  is equivalent to  $\|\sigma_q^i\|_{L^2}$  if  $q_1 < q \leq q_0$ , we can use  $\|\Lambda^{-1} \sigma_q^i\|_{L^2}$  instead of  $\|\sigma_q^i\|_{L^2}$  to fill up the gap. To do so, applying the operator  $\Lambda^{-1}$  to (3.2)<sub>2</sub>, and then taking the  $L^2$  inner product of the resulting equation with  $\Lambda^{-1} \sigma_q^i$ , we obtain

$$\begin{aligned}
 &\frac{1}{2} \frac{d}{dt} \|\Lambda^{-1} \sigma_q^i\|_{L^2}^2 + \frac{1}{\text{We}} \|\Lambda^{-1} \sigma_q^i\|_{L^2}^2 + \frac{\omega}{\text{We}} (u_q^i | \sigma_q^i) \\
 (3.34) \quad &= (\Lambda^{-1} \tilde{h}_q^i | \Lambda^{-1} \sigma_q^i) - (\Lambda^{-1} (\dot{S}_{q-1} u \cdot \nabla \sigma_q^i) | \Lambda^{-1} \sigma_q^i).
 \end{aligned}$$

Multiplying (3.34) by  $\frac{\text{We}}{\omega\text{Re}}$ , then adding the resulting equation to (3.6), the cross term  $(u_q^i | \sigma_q^i)$  is canceled, and we arrive at

$$\begin{aligned}
 &\frac{1}{2} \frac{d}{dt} \left( \|u_q^i\|_{L^2}^2 + \frac{\text{We}}{\omega\text{Re}} \|\Lambda^{-1} \sigma_q^i\|_{L^2}^2 \right) + \frac{1-\omega}{\text{Re}} \|\Lambda u_q^i\|_{L^2}^2 + \frac{1}{\omega\text{Re}} \|\Lambda^{-1} \sigma_q^i\|_{L^2}^2 \\
 (3.35) \quad &\leq (\tilde{f}_q^i | u_q^i) + \frac{\text{We}}{\omega\text{Re}} \left( (\Lambda^{-1} \tilde{h}_q^i | \Lambda^{-1} \sigma_q^i) - (\Lambda^{-1} (\dot{S}_{q-1} u \cdot \nabla \sigma_q^i) | \Lambda^{-1} \sigma_q^i) \right),
 \end{aligned}$$

In view of (2.4) and (3.20), it is not difficult to verify that

$$\begin{aligned}
 \frac{1-\omega}{\text{Re}} \|\Lambda u_q^i\|_{L^2}^2 + \frac{1}{\omega \text{Re}} \|\Lambda^{-1} \sigma_q^i\|_{L^2}^2 &\geq \frac{1-\omega}{\text{Re}} \left( \frac{3}{4} 2^{q_1} \right)^2 \|u_q^i\|_{L^2}^2 + \frac{1}{\omega \text{Re}} \|\Lambda^{-1} \sigma_q^i\|_{L^2}^2 \\
 &\geq \frac{3^4 \text{We}(1-\omega)}{2^{16} (\text{Re} + \text{We})^2} \|u_q^i\|_{L^2}^2 + \frac{1}{\omega \text{Re}} \|\Lambda^{-1} \sigma_q^i\|_{L^2}^2 \\
 (3.36) \quad &\geq \frac{3^4 \text{We}(1-\omega)}{2^{16} (\text{Re} + \text{We})^2} \left( \|u_q^i\|_{L^2}^2 + \frac{\text{We}}{\omega \text{Re}} \|\Lambda^{-1} \sigma_q^i\|_{L^2}^2 \right).
 \end{aligned}$$

Using the divergence free condition  $\text{div} u = 0$ , one easily deduces that

$$(\Lambda^{-1}(\dot{S}_{q-1} u \cdot \nabla \sigma_q^i) | \Lambda^{-1} \sigma_q^i) = ([\Lambda^{-1}, \dot{S}_{q-1} u \cdot \nabla] \sigma_q^i) | \Lambda^{-1} \sigma_q^i,$$

then Lemma 2.2 and Lemma 2.3 imply

$$(3.37) \quad |(\Lambda^{-1}(\dot{S}_{q-1} u \cdot \nabla \sigma_q^i) | \Lambda^{-1} \sigma_q^i)| \leq C \|\nabla \dot{S}_{q-1} u\|_{L^\infty} \|\Lambda^{-1} \sigma_q^i\|_{L^2}^2 \leq C \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|\Lambda^{-1} \sigma_q^i\|_{L^2}^2.$$

Substituting (3.36) and (3.37) into (3.35), we find that there exists a constant  $C$  depending on  $d$ , but independent of  $\text{Re}$ ,  $\text{We}$ , and  $\omega$ , such that if  $q_1 < q \leq q_0$ , there holds

$$(3.38) \quad \frac{d}{dt} (Y_q^i)^2 + \frac{3^4 \text{We}(1-\omega)}{2^{16} (\text{Re} + \text{We})^2} (Y_q^i)^2 \leq C Y_q^i \left( \|\tilde{f}_q^i\|_{L^2} + \sqrt{\frac{\text{We}}{\omega \text{Re}}} \|\Lambda^{-1} \tilde{h}_q^i\|_{L^2} + \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} Y_q^i \right).$$

### The smoothing effect of $u$ .

Now we are in a position to show the smoothing effect of the velocity  $u$ . In fact, in the low frequency part, the smoothing effect of  $u_q$  has been obtained in (3.24). More precisely, a direct consequence of (3.24) and (3.14) gives

$$\begin{aligned}
 &\|u_q^i(t)\|_{L^2} + \|\sigma_q^i(t)\|_{L^2} + 2^{2q} \left( \|u_q^i\|_{L_t^1(L^2)} + \|\sigma_q^i\|_{L_t^1(L^2)} \right) \\
 &\leq C \left( \|u_q^i(0)\|_{L^2} + \|\sigma_q^i(0)\|_{L^2} + \|\tilde{f}_q^i\|_{L_t^1(L^2)} + \|\tilde{h}_q^i\|_{L_t^1(L^2)} \right. \\
 (3.39) \quad &\left. + \int_0^t \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \left( \|u_q^i\|_{L^2} + \|\sigma_q^i\|_{L^2} \right) dt' \right),
 \end{aligned}$$

for  $q \leq q_1$ . Nevertheless, in the high frequency part, we infer from (3.33), (3.27) and (3.38) that

$$\begin{aligned}
 &\|u_q^i(t)\|_{L^2} + \left\| \frac{(1-\omega)\text{We}}{\omega \text{Re}} \sigma_q^i(t) \right\|_{L^2} + \frac{1}{(1-\omega)\text{We}} \left( \|u_q^i\|_{L_t^1(L^2)} + \left\| \frac{(1-\omega)\text{We}}{\omega \text{Re}} \sigma_q^i \right\|_{L_t^1(L^2)} \right) \\
 (3.40) \quad &\leq C \left( \|u_q^i(0)\|_{L^2} + \left\| \frac{(1-\omega)\text{We}}{\omega \text{Re}} \sigma_q^i(0) \right\|_{L^2} + \|\tilde{f}_q^i\|_{L_t^1(L^2)} + \left\| \frac{(1-\omega)\text{We}}{\omega \text{Re}} \tilde{h}_q^i \right\|_{L_t^1(L^2)} \right),
 \end{aligned}$$

for  $q > q_0$ , and

$$\begin{aligned}
 &\|u_q^i(t)\|_{L^2} + \sqrt{\frac{\text{We}}{\omega \text{Re}}} \|\Lambda^{-1} \sigma_q^i(t)\|_{L^2} + \frac{3^4 \text{We}(1-\omega)}{2^{16} (\text{Re} + \text{We})^2} \left( \|u_q^i\|_{L_t^1(L^2)} + \sqrt{\frac{\text{We}}{\omega \text{Re}}} \|\Lambda^{-1} \sigma_q^i\|_{L_t^1(L^2)} \right) \\
 &\leq C \left( \|u_q^i(0)\|_{L^2} + \sqrt{\frac{\text{We}}{\omega \text{Re}}} \|\Lambda^{-1} \sigma_q^i(0)\|_{L^2} + \|\tilde{f}_q^i\|_{L_t^1(L^2)} + \sqrt{\frac{\text{We}}{\omega \text{Re}}} \|\Lambda^{-1} \tilde{h}_q^i\|_{L_t^1(L^2)} \right. \\
 (3.41) \quad &\left. + \int_0^t \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \left( \|u_q^i\|_{L^2} + \sqrt{\frac{\text{We}}{\omega \text{Re}}} \|\Lambda^{-1} \sigma_q^i\|_{L^2} \right) dt' \right),
 \end{aligned}$$

for  $q_1 < q \leq q_0$ . Recalling the definitions of  $q_1$  and  $q_0$ , and using (2.4) again, one deduces that

$$(3.42) \quad \|\sigma_q^i\|_{L^2} \leq \frac{8}{3} 2^{q_0} \|\Lambda^{-1} \sigma_q^i\|_{L^2} \leq \frac{2^7}{3^2(1-\omega)} \sqrt{\frac{\text{Re}}{\text{We}}} \|\Lambda^{-1} \sigma_q^i\|_{L^2},$$

and

$$(3.43) \quad \|\sigma_q^i\|_{L^2} \geq \frac{3}{4} 2^{q_1} \|\Lambda^{-1} \sigma_q^i\|_{L^2} \geq \frac{3^2}{2^8} \frac{\sqrt{\text{ReWe}}}{\text{Re} + \text{We}} \|\Lambda^{-1} \sigma_q^i\|_{L^2}.$$

By virtue of (3.42), (3.41) implies

$$(3.44) \quad \begin{aligned} & \|u_q^i(t)\|_{L^2} + \frac{3^2(1-\omega)\text{We}}{2^7 \sqrt{\omega}\text{Re}} \|\sigma_q^i(t)\|_{L^2} \\ & + \frac{3^4\text{We}(1-\omega)}{2^{16}(\text{Re} + \text{We})^2} \|u_q^i\|_{L_t^1(L^2)} + \frac{3^6(\text{We})^2(1-\omega)^2}{\sqrt{\omega}2^{23}(\text{Re} + \text{We})^2} \frac{1}{\text{Re}} \|\sigma_q^i\|_{L_t^1(L^2)} \\ & \leq C \left( \|u_q^i(0)\|_{L^2} + \sqrt{\frac{\text{We}}{\omega\text{Re}}} \|\Lambda^{-1} \sigma_q^i(0)\|_{L^2} + \|\tilde{f}_q^i\|_{L_t^1(L^2)} + \sqrt{\frac{\text{We}}{\omega\text{Re}}} \|\Lambda^{-1} \tilde{h}_q^i\|_{L_t^1(L^2)} \right. \\ & \quad \left. + \int_0^t \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \left( \|u_q^i\|_{L^2} + \sqrt{\frac{\text{We}}{\omega\text{Re}}} \|\Lambda^{-1} \sigma_q^i\|_{L^2} \right) dt' \right), \end{aligned}$$

for  $q_1 < q \leq q_0$ . It can be seen from (3.40) and (3.44) that we have not yet obtain the soothing effect of  $u_q$  if  $q > q_1$ . Fortunately, the damping effect of  $\sigma_q$  is obtained both in (3.40) and (3.44). This enables us to get the smoothing effect of  $u_q$  by means of the equation of the velocity  $u$ . To this end, taking the  $L^2$  inner product of (3.2)<sub>1</sub> with  $u_q^i$ , integrating by parts, we have

$$\frac{1}{2} \frac{d}{dt} \|u_q^i\|_{L^2}^2 + \frac{1-\omega}{\text{Re}} \|\Lambda u_q^i\|_{L^2}^2 = \frac{1}{\text{Re}} (\sigma_q^i | u_q^i) + (\tilde{f}_q^i | u_q^i).$$

Using the left hand side of (2.4), we easily get

$$(3.45) \quad \|u_q^i(t)\|_{L^2} + \frac{1-\omega}{\text{Re}} \left( \frac{3}{4} 2^q \right)^2 \|u_q^i\|_{L_t^1(L^2)} \leq \|u_q^i(0)\|_{L^2} + \frac{1}{\text{Re}} \|\sigma_q^i\|_{L_t^1(L^2)} + \|\tilde{f}_q^i\|_{L_t^1(L^2)}.$$

Multiplying (3.40) by  $2\omega$ , and adding the resulting equation to (3.45), we arrive at

$$(3.46) \quad \begin{aligned} & \|u_q^i(t)\|_{L^2} + \frac{2(1-\omega)\text{We}}{\text{Re}} \|\sigma_q^i(t)\|_{L^2} + \frac{1-\omega}{\text{Re}} \left( \frac{3}{4} 2^q \right)^2 \|u_q^i\|_{L_t^1(L^2)} + \frac{1}{\text{Re}} \|\sigma_q^i\|_{L_t^1(L^2)} \\ & \leq C \left( \|u_q^i(0)\|_{L^2} + \|\sigma_q^i(0)\|_{L^2} + \|\tilde{f}_q^i\|_{L_t^1(L^2)} + \|\tilde{h}_q^i\|_{L_t^1(L^2)} \right), \end{aligned}$$

for  $q > q_0$ . Multiplying (3.44) by  $\frac{\sqrt{\omega}2^{24}(\text{Re}+\text{We})^2}{3^6(\text{We})^2(1-\omega)^2}$ , and adding the resulting equation to (3.45), using (3.43), we obtain

$$(3.47) \quad \begin{aligned} & \|u_q^i(t)\|_{L^2} + \frac{2^{17}(\text{Re} + \text{We})^2}{3^4\text{ReWe}(1-\omega)} \|\sigma_q^i(t)\|_{L^2} + \frac{1-\omega}{\text{Re}} \left( \frac{3}{4} 2^q \right)^2 \|u_q^i\|_{L_t^1(L^2)} + \frac{1}{\text{Re}} \|\sigma_q^i\|_{L_t^1(L^2)} \\ & \leq \frac{C}{(1-\omega)^2} \left( \|u_q^i(0)\|_{L^2} + \|\sigma_q^i(0)\|_{L^2} + \|\tilde{f}_q^i\|_{L_t^1(L^2)} \right. \\ & \quad \left. + \|\tilde{h}_q^i\|_{L_t^1(L^2)} + \int_0^t \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \left( \|u_q^i\|_{L^2} + \|\sigma_q^i\|_{L^2} \right) dt' \right), \end{aligned}$$

for  $q_1 < q \leq q_0$ . Now collecting (3.46), (3.47), and (3.39), we conclude that

$$(3.48) \quad \begin{aligned} & \|u^i\|_{\dot{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} + (1-\omega) \|\sigma^i\|_{\dot{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} + (1-\omega) \|u^i\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} + \|\sigma^i\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1, \frac{d}{2}-1})} \\ & \leq \frac{C}{(1-\omega)^2} \left( \|u_0^i\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|\sigma_0^i\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \int_0^t \sum_{q \in \mathbb{Z}} 2^{q(\frac{d}{2}-1)} \left( \|\tilde{f}_q^i\|_{L^2} + \|\tilde{h}_q^i\|_{L^2} \right) dt' \right. \\ & \quad \left. + \int_0^t \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \left( \|u^i\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|\sigma^i\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \right) dt' \right), \end{aligned}$$

with some positive constant  $C$  depending on  $d, \text{Re}$  and  $\text{We}$ , but independent of  $\omega$ .

**Estimates of  $\sum_{q \in \mathbb{Z}} 2^{q(\frac{d}{2}-1)} (\|\tilde{f}_q^i\|_{L^2} + \|\tilde{h}_q^i\|_{L^2})$ .**

Recalling that  $\tilde{f}_q^i := \dot{S}_{q-1} u \cdot \nabla u_q^i + f_q^i$ , and  $f^i := -(\mathbb{P}(u \cdot \nabla)u)^i = -(\delta_{im} + \partial_i \Lambda^{-2} \partial_m) u^k \partial_k u^m$ , in order to estimate  $\sum_{q \in \mathbb{Z}} 2^{q(\frac{d}{2}-1)} \|\tilde{f}_q^i\|_{L^2}$ , it suffices to bound  $\sum_{q \in \mathbb{Z}} 2^{q(\frac{d}{2}-1)} \|\dot{S}_{q-1} u^k \partial_k u_q^i\|_{L^2}$  and  $\|(\delta_{im} + \partial_i \Lambda^{-2} \partial_m) u^k \partial_k u^m\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}$ .

*Estimates of  $\sum_{q \in \mathbb{Z}} 2^{q(\frac{d}{2}-1)} \|\dot{S}_{q-1} u^k \partial_k u_q^i\|_{L^2}$ .*

- Case 1:  $1 \leq i \leq d-1$ .

$$\begin{aligned} \sum_{q \in \mathbb{Z}} 2^{q(\frac{d}{2}-1)} \|\dot{S}_{q-1} u^k \partial_k u_q^i\|_{L^2} &\leq C \|\dot{S}_{q-1} u\|_{L^\infty} \sum_{q \in \mathbb{Z}} 2^{\frac{d}{2}} \|u_q^h\|_{L^2} \\ &\leq C \|u\|_{L^\infty} \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \leq C \left( \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 + \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right). \end{aligned}$$

- Case 2:  $i = d$ .

$$\begin{aligned} \sum_{q \in \mathbb{Z}} 2^{q(\frac{d}{2}-1)} \|\dot{S}_{q-1} u^k \partial_k u_q^d\|_{L^2} &\leq C \sum_{q \in \mathbb{Z}} 2^{q(\frac{d}{2}-1)} \|\dot{S}_{q-1} u^h \partial_h u_q^d\|_{L^2} + \sum_{q \in \mathbb{Z}} 2^{q(\frac{d}{2}-1)} \|\dot{S}_{q-1} u^d \partial_d u_q^d\|_{L^2} \\ &\leq C \|u^h\|_{L^\infty} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + C \|u^d\|_{L^\infty} \|\text{div}_h u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \\ &\leq C \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}}}. \end{aligned}$$

Combining these two estimates, we are led to

$$(3.49) \quad \sum_{q \in \mathbb{Z}} 2^{q(\frac{d}{2}-1)} \|\dot{S}_{q-1} u^k \partial_k u_q^i\|_{L^2} \leq C \left( \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 + \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) \quad \text{for } 1 \leq i \leq d.$$

*Estimates of  $\|(\delta_{im} + \partial_i \Lambda^{-2} \partial_m) u^k \partial_k u^m\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}$ .*

Using Proposition 2.2, we have

$$\begin{aligned} &\|(\delta_{im} + \partial_i \Lambda^{-2} \partial_m) \dot{T}_{u^k} \partial_k u^m\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \\ &\leq \sum_{1 \leq m, k \leq d-1} \|(\delta_{im} + \partial_i \Lambda^{-2} \partial_m) \dot{T}_{u^k} \partial_k u^m\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \sum_{1 \leq k \leq d-1} \|(\delta_{id} + \partial_i \Lambda^{-2} \partial_d) \dot{T}_{u^k} \partial_k u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \\ &\quad + \sum_{1 \leq m \leq d-1} \|(\delta_{im} + \partial_i \Lambda^{-2} \partial_m) \dot{T}_{u^d} \partial_d u^m\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|(\delta_{id} + \partial_i \Lambda^{-2} \partial_d) \dot{T}_{u^d} \partial_d u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \\ &\leq C \left( \|u^h\|_{L^\infty} \|\partial_h u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|u^h\|_{L^\infty} \|\partial_h u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|u^d\|_{L^\infty} \|\partial_d u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|u^d\|_{L^\infty} \|\text{div}_h u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \right) \\ &\leq C \left( \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 + \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right), \end{aligned}$$

and

$$\begin{aligned} &\|(\delta_{im} + \partial_i \Lambda^{-2} \partial_m) \dot{T}'_{\partial_k u^m} u^k\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \\ &\leq \sum_{1 \leq m, k \leq d-1} \|(\delta_{im} + \partial_i \Lambda^{-2} \partial_m) \dot{T}'_{\partial_k u^m} u^k\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \sum_{1 \leq k \leq d-1} \|(\delta_{id} + \partial_i \Lambda^{-2} \partial_d) \dot{T}'_{\partial_k u^d} u^k\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \\ &\quad + \sum_{1 \leq m \leq d-1} \|(\delta_{im} + \partial_i \Lambda^{-2} \partial_m) \dot{T}'_{\partial_d u^m} u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|(\delta_{id} + \partial_i \Lambda^{-2} \partial_d) \dot{T}'_{\partial_d u^d} u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \end{aligned}$$



$$\begin{aligned}
 &\leq C \left( \|\partial_h u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|\partial_h u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|\partial_d u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|\operatorname{div}_h u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) \\
 &\leq C \left( \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right).
 \end{aligned}$$

Therefore, according to Bony's decomposition, there holds

$$(3.50) \quad \|(\delta_{im} + \partial_i \Lambda^{-2} \partial_m) u^k \partial_k u^m\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \leq C \left( \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) \quad \text{for } 1 \leq i \leq d.$$

It follows from (3.49), (3.50) and Proposition 2.1 that

$$\begin{aligned}
 \sum_{q \in \mathbb{Z}} 2^{q(\frac{d}{2}-1)} \|\tilde{f}_q^i\|_{L^2} &\leq C \left( \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) \\
 (3.51) \quad &\leq C \left( \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{\frac{1}{2}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^{\frac{1}{2}} \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^{\frac{1}{2}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \right) \quad \text{for } 1 \leq i \leq d.
 \end{aligned}$$

Next, we turn to estimate  $\sum_{q \in \mathbb{Z}} 2^{q(\frac{d}{2}-1)} \|\tilde{h}_q^i\|_{L^2}$ . Noticing that

$$\begin{aligned}
 \tilde{h}_q^i &= \dot{S}_{q-1} u \cdot \nabla \sigma_q^i + h_q^i \\
 &= \dot{S}_{q-1} u \cdot \nabla \sigma_q^i - \dot{\Delta}_q (\mathbb{P} \operatorname{div} (u \cdot \nabla) \tau + \mathbb{P} \operatorname{div} g_\alpha(\tau, \nabla u))^i \\
 &= \left[ \dot{S}_{q-1} u \cdot \nabla \sigma_q^i - \dot{\Delta}_q (\mathbb{P} \dot{T}_{u^k} \partial_k \operatorname{div} \tau)^i \right] - \dot{\Delta}_q (\delta_{im} + \partial_i \Lambda^{-2} \partial_m) \dot{T}_{\partial_l u^k} \partial_k \tau^{l,m} \\
 &\quad - \dot{\Delta}_q (\delta_{im} + \partial_i \Lambda^{-2} \partial_m) \partial_l \dot{T}'_{\partial_k \tau^{l,m}} u^k - \dot{\Delta}_q (\mathbb{P} \operatorname{div} g_\alpha(\tau, \nabla u))^i,
 \end{aligned}$$

we bound  $\sum_{q \in \mathbb{Z}} 2^{q(\frac{d}{2}-1)} \|\tilde{h}_q^i\|_{L^2}$  term by term. First of all, thanks to the commutator estimate Lemma 10.25 in [1], we find that

$$\begin{aligned}
 &\sum_{q \in \mathbb{Z}} 2^{q(\frac{d}{2}-1)} \left\| \left[ \dot{S}_{q-1} u \cdot \nabla \sigma_q^i - \dot{\Delta}_q (\mathbb{P} \dot{T}_{u^k} \partial_k \operatorname{div} \tau)^i \right] \right\|_{L^2} \\
 (3.52) \quad &\leq C \|\nabla u\|_{L^\infty} \sum_{q \in \mathbb{Z}} \sum_{|q'-q| \leq 4} 2^{q(\frac{d}{2}-1)} \|\dot{\Delta}_{q'} \operatorname{div} \tau\|_{L^2} \leq C \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.
 \end{aligned}$$

The second term contained in  $\sum_{q \in \mathbb{Z}} 2^{q(\frac{d}{2}-1)} \|\tilde{h}_q^i\|_{L^2}$  is nothing but  $\|(\delta_{im} + \partial_i \Lambda^{-2} \partial_m) \dot{T}_{\partial_l u^k} \partial_k \tau^{l,m}\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}$ , the estimate of which is easier. Indeed, using Proposition 2.1 and 2.2 directly yields

$$\|(\delta_{im} + \partial_i \Lambda^{-2} \partial_m) \dot{T}_{\partial_l u^k} \partial_k \tau^{l,m}\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \leq C \|\nabla u\|_{L^\infty} \|\nabla \tau\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \leq C \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.$$

Now we go to bound the third term,

$$\|(\delta_{im} + \partial_i \Lambda^{-2} \partial_m) \partial_l \dot{T}'_{\partial_k \tau^{l,m}} u^k\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \leq C \|\dot{T}'_{\partial_k \tau} u^k\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \leq C \|\dot{T}'_{\partial_h \tau} u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + C \|\dot{T}'_{\partial_d \tau} u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.$$

Using Proposition 2.2 once more and Bernstein's inequality, it is easy to see that

$$(3.53) \quad \|\dot{T}'_{\partial_h \tau} u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \leq C \|\partial_h \tau\|_{\dot{B}_{\infty,\infty}^{-1}} \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \leq C \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.$$

However, the estimate of  $\|\dot{T}'_{\partial_d \tau} u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}}}$  is a little bit complicated. In fact, using Lemma 2.3 and Bernstein's inequality, we have

$$\begin{aligned}
 \|\dot{T}'_{\partial_d \tau} u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}}} &= \sum_{q \in \mathbb{Z}} 2^{q\frac{d}{2}} \|\dot{\Delta}_q \dot{T}'_{\partial_d \tau} u^d\|_{L^2} \\
 &\leq \sum_{q \in \mathbb{Z}} \sum_{q'-q' \leq 1} 2^{q\frac{d}{2}} \|\dot{\Delta}_q (\dot{S}_{q'+2} \partial_d \tau \dot{\Delta}_{q'} u^d)\|_{L^2}
 \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{q \in \mathbb{Z}} \sum_{q-q' \leq 1} 2^{q\frac{d}{2}} \|\dot{S}_{q'+2} \partial_d \tau\|_{L_{x_h}^\infty(L_{x_d}^2)} \|\dot{\Delta}_{q'} u^d\|_{L_{x_h}^2(L_{x_d}^\infty)} \\
&\leq C \sum_{q \in \mathbb{Z}} \sum_{q-q' \leq 1} \sum_{q'' \leq q'+1} 2^{q\frac{d}{2}} \|\dot{\Delta}_{q''} \partial_d \tau\|_{L_{x_h}^\infty(L_{x_d}^2)} \|\dot{\Delta}_{q'} u^d\|_{L^2}^{\frac{1}{2}} \|\dot{\Delta}_{q'} \operatorname{div}_h u^h\|_{L^2}^{\frac{1}{2}} \\
&\leq C \sum_{q \in \mathbb{Z}} \sum_{q-q' \leq 1} \sum_{q'' \leq q'+1} 2^{q\frac{d}{2}} 2^{q''\frac{d+1}{2}} \|\dot{\Delta}_{q''} \tau\|_{L^2} \|\dot{\Delta}_{q'} u^d\|_{L^2}^{\frac{1}{2}} \|\dot{\Delta}_{q'} \operatorname{div}_h u^h\|_{L^2}^{\frac{1}{2}} \\
&\leq C \sum_{q \in \mathbb{Z}} \sum_{q-q' \leq 1} 2^{(q-q')\frac{d}{2}} \sum_{q'' \leq q'+1} 2^{(q''-q')\frac{1}{2}} \left( 2^{q''\frac{d}{2}} \|\dot{\Delta}_{q''} \tau\|_{L^2} \right) \\
&\quad \times \left( 2^{q'\frac{d}{2}+1} \|\dot{\Delta}_{q'} u^d\|_{L^2} \right)^{\frac{1}{2}} \left( 2^{q'\frac{d}{2}} \|\dot{\Delta}_{q'} \operatorname{div}_h u^h\|_{L^2} \right)^{\frac{1}{2}} \\
&\leq C \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.
\end{aligned}$$

Finally, we would like to point out that the estimate of the term  $\|(\mathbb{P} \operatorname{div}_\alpha(\tau, \nabla u))^i\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}$  is somehow “bad” due to the appearance of  $\sum_{1 \leq k \leq d-1} \|\dot{T}_{\tau^i, k} \partial_k u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}}}$  both for horizontal and vertical direction. Thus, there is no need to bound  $\|(\mathbb{P} \operatorname{div}_\alpha(\tau, \nabla u))^i\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}$  from different directions. In view of Proposition 2.1 and Corollary 2.2, we easily get

$$\begin{aligned}
\|(\mathbb{P} \operatorname{div}_\alpha(\tau, \nabla u))^i\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} &\leq C \|g_\alpha(\tau, \nabla u)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \\
(3.54) \quad &\leq C \|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \leq C \left( \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right) \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.
\end{aligned}$$

Combining the above estimates, we find that

$$(3.55) \quad \sum_{q \in \mathbb{Z}} 2^{q(\frac{d}{2}-1)} \|\tilde{h}_q^i\|_{L^2} \leq C \left( \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right) \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.$$

Substituting (3.51) and (3.55) into (3.48) yields

$$\begin{aligned}
&\|u^i\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + (1-\omega) \|\sigma^i\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + (1-\omega) \|u^i\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} + \|\sigma^i\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1, \frac{d}{2}-1})} \\
&\leq \frac{C}{(1-\omega)^2} \left( \|u_0^i\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|\sigma_0^i\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \int_0^t \left( \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right) \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}}} dt' \right. \\
&\quad + \int_0^t \left( \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^{\frac{1}{2}} \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^{\frac{1}{2}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \right) dt' \\
&\quad \left. + \int_0^t \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \left( \|u^i\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|\sigma^i\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \right) dt' \right) \\
&\leq \frac{C}{(1-\omega)^2} \left( \|u_0^i\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|\sigma_0^i\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \int_0^t \left( \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right) \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}}} dt' \right. \\
&\quad + \int_0^t \left( \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^{\frac{1}{2}} \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}^{\frac{1}{2}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \right) dt' \\
&\quad \left. + \int_0^t \left( \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^{\frac{1}{2}} \right) \|u^i\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} dt' \right) \quad \text{for all } i \in \{1, 2, \dots, d\}.
\end{aligned}$$

Consequently, there exist two constants  $C_1$  and  $C_2$  depending on  $d$ ,  $\text{Re}$  and  $\text{We}$ , but independent of  $\omega$ , such that (3.3) and (3.4) hold.

**The damping effect of  $\tau$ .**

In order to deal with the nonlinear coupling between  $u$  and  $\tau$ , it is necessary to take advantage of the damping effect of  $\tau$ . To do so, applying  $\dot{\Delta}_q$  to the equation of  $\tau$  yields

$$\partial_t \tau_q + \dot{S}_{q-1} u \cdot \nabla \tau_q + \frac{\tau_q}{\text{We}} = \frac{2\omega}{\text{We}} D(u_q) + \dot{S}_{q-1} u \cdot \nabla \tau_q - \dot{\Delta}_q(u \cdot \nabla \tau) - \dot{\Delta}_q g_\alpha(\tau, \nabla u).$$

Taking the inner product of the above equation with  $\tau_q$ , integrating by parts, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\tau_q\|_{L^2}^2 + \frac{1}{\text{We}} \|\tau_q\|_{L^2}^2 \\ &= \frac{2\omega}{\text{We}} (D(u_q) | \tau_q) + (\dot{S}_{q-1} u \cdot \nabla \tau_q - \dot{\Delta}_q(u \cdot \nabla \tau) | \tau_q) - (\dot{\Delta}_q g_\alpha(\tau, \nabla u) | \tau_q). \end{aligned}$$

It follows that

$$\begin{aligned} \|\tau_q(t)\|_{L^2} + \frac{1}{\text{We}} \|\tau_q\|_{L_t^1(L^2)} &\leq C \left( \|\tau_q(0)\|_{L^2} + \omega 2^q \|u_q\|_{L^2} + \|\dot{S}_{q-1} u \cdot \nabla \tau_q - \dot{\Delta}_q(\dot{T}_{u^k} \partial_k \tau)\|_{L^2} \right. \\ &\quad \left. + \|\dot{\Delta}_q \dot{T}'_{\partial_k \tau} u^k\|_{L^2} + \|\dot{\Delta}_q g_\alpha(\tau, \nabla u)\|_{L^2} \right). \end{aligned}$$

Multiplying the above equation by  $2^{q\frac{d}{2}}$ , and summing over  $q \in \mathbb{Z}$ , we find that there exists a constant  $C_3$  depending on  $d$ ,  $\text{Re}$  and  $\text{We}$ , but independent of  $\omega$ , such that

$$\begin{aligned} & \|\tau\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} + \|\tau\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})} \\ &\leq C_3 \left( \|\tau_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \omega \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} + \int_0^t \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}}} dt' \right) \\ (3.56) \quad &\leq C_3 \left( \|\tau_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \omega \left( \|u^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} + \|u^d\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \right) + \int_0^t (\|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}) \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}}} dt' \right), \end{aligned}$$

where we have used the following nonlinear estimates which can be obtained in a similar manner to those of (3.52), (3.53) and (3.54),

$$\sum_{q \in \mathbb{Z}} 2^{\frac{d}{2}q} \|\dot{S}_{q-1} u \cdot \nabla \tau_q - \dot{\Delta}_q(\dot{T}_{u^k} \partial_k \tau)\|_{L^2} \leq C \|\nabla u\|_{L^\infty} \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}}},$$

and

$$\|\dot{T}'_{\partial_k \tau} u^k\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \leq C \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}}}, \quad \|g_\alpha(\tau, \nabla u)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \leq C \|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}.$$

This completes the proof Proposition 3.1.  $\square$

**Remark 3.1.** We can bound  $\sum_{q \in \mathbb{Z}} 2^{q(\frac{d}{2}-1)} \|\tilde{h}_q^i\|_{L^2}$  without making a distinction between horizontal direction and vertical direction due to the appearance of the nonlinear term  $g_\alpha(\tau, \nabla u)$ . Our above estimates of  $\sum_{q \in \mathbb{Z}} 2^{q(\frac{d}{2}-1)} \|\tilde{h}_q^i\|_{L^2}$  aim to reveal that the convection term  $u \cdot \nabla \tau$  behaves better than  $g_\alpha(\tau, \nabla u)$ .

The rest part of this section is to close the estimates (3.3)–(3.5) in Proposition 3.1. Before proceeding any further, let us denote

$$\begin{aligned} A_0^d &:= \frac{4C_2}{(1-\omega)^3} (A^d(0) + 1), \\ A_0^h &:= \frac{16C_1C_2C_3}{(1-\omega)^6} \exp\left(\frac{8(C_1^2 + C_3)}{(1-\omega)^6} (A_0^d)^2\right) (A^h(0) + \omega A^d(0) + B(0)), \\ B_0 &:= \frac{192C_1C_2^2C_3^2}{(1-\omega)^6} \exp\left(\frac{12(C_1^2 + C_3)}{(1-\omega)^6} (A_0^d)^2\right) (A^h(0) + \omega A^d(0) + B(0)). \end{aligned}$$

**Proposition 3.2.** Assume that  $(u, \tau)$  is a solution of system (1.1) on  $[0, T]$ , satisfying

$$(3.57) \quad A^h(T) \leq A_0^h, \quad A^d(T) \leq A_0^d, \quad B(T) \leq B_0.$$

If

$$(3.58) \quad \frac{3}{2}A_0^h + \sqrt{A_0^h A_0^d} + B_0 \leq \frac{(1-\omega)^3}{2C_2}, \quad A_0^h(A_0^h + B_0) \leq 1,$$

$$(3.59) \quad B_0 \left( 1 + \frac{2C_2}{(1-\omega)^3} (A_0^h + B_0) \right) \leq \frac{(1-\omega)^3}{4C_1},$$

$$(3.60) \quad \frac{3}{2}A_0^h \leq 1, \quad \frac{2C_2 C_3}{(1-\omega)^3} A_0^h \leq \frac{1}{2},$$

then there hold

$$(3.61) \quad A^h(T) \leq \frac{1}{2}A_0^h, \quad A^d(T) \leq \frac{1}{2}A_0^d, \quad \text{and} \quad B(T) \leq \frac{1}{2}B_0.$$

*Proof.* From (3.4), it is easy to verify that

$$(3.62) \quad \begin{aligned} A^d(t) &\leq \frac{C_2}{(1-\omega)^3} \left( A^d(0) + A^h(t) (A^h(t) + B(t)) + \frac{1}{2} A^h(t) \|u^d\|_{\tilde{L}_t^\infty(B_{2,1}^{\frac{d}{2}-1})} \right. \\ &\quad \left. + \frac{1}{2} A^h(t) \|u^d\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} + \left( A^h(t) + \sqrt{A^h(t) A^d(t)} \right) \|u^d\|_{\tilde{L}_t^\infty(B_{2,1}^{\frac{d}{2}-1})} + B(t) \|u^d\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \right) \\ &\leq \frac{C_2}{(1-\omega)^3} \left( A^d(0) + A^h(t) (A^h(t) + B(t)) + \left( \frac{3}{2} A^h(t) + \sqrt{A^h(t) A^d(t)} + B(t) \right) A^d(t) \right). \end{aligned}$$

If

$$(3.63) \quad \frac{3}{2}A_0^h + \sqrt{A_0^h A_0^d} + B_0 \leq \frac{(1-\omega)^3}{2C_2},$$

then it follows from (3.57) and (3.62) that

$$(3.64) \quad A^d(t) \leq \frac{2C_2}{(1-\omega)^3} (A^d(0) + A^h(t) (A^h(t) + B(t))), \quad t \in [0, T].$$

Furthermore, if

$$(3.65) \quad A_0^h (A_0^h + B_0) \leq 1,$$

then (3.64) changes to be

$$(3.66) \quad A^d(t) \leq \frac{2C_2}{(1-\omega)^3} (A^d(0) + 1) = \frac{1}{2}A_0^d, \quad t \in [0, T].$$

Next we go to bound  $B(t)$ . To do this, combining (3.56) and (3.64), we are led to

$$(3.67) \quad \begin{aligned} B(t) &\leq C_3 \left( B(0) + \omega A_0^h + \omega A^d(t) + \int_0^t (\|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}) \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}}} dt' \right) \\ &\leq C_3 \left( B(0) + \omega A_0^h + \frac{2C_2 \omega}{(1-\omega)^3} (A^d(0) + (A^h(t))^2 + A^h(t) B(t)) \right. \\ &\quad \left. + \int_0^t (\|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}) \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}}} dt' \right) \\ &\leq C_3 \left( B(0) + \frac{3}{2} \omega A_0^h + \frac{2C_2 \omega}{(1-\omega)^3} A^d(0) + \int_0^t (\|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}) \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}}} dt' \right) + \frac{1}{2} B(t), \end{aligned}$$

provided

$$(3.68) \quad \frac{2C_2 C_3}{(1-\omega)^3} A_0^h \leq \frac{1}{2}.$$

Using Gronwall's inequality, we obtain

$$(3.69) \quad B(t) \leq 2C_3 \exp\left(2C_3(1 + A_0^d)\right) \left(B(0) + \frac{3}{2}\omega A_0^h + \frac{2C_2\omega}{(1-\omega)^3} A^d(0)\right) \leq \tilde{B}_0,$$

where  $\tilde{B}_0$  is defined to be

$$(3.70) \quad \tilde{B}_0 := 2C_3 \exp\left(2C_3(1 + A_0^d)\right) \left(B(0) + \frac{3}{2}\omega + \frac{2C_2\omega}{(1-\omega)^3} A^d(0)\right).$$

To estimate  $A^h(t)$ , we infer from (3.3) that

$$(3.71) \quad \begin{aligned} A^h(t) &\leq \frac{C_1}{(1-\omega)^3} \left( A^h(0) + \int_0^t \left( \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \frac{C_1}{(1-\omega)^3} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right) \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} dt' \right. \\ &\quad + \frac{(1-\omega)^3}{4C_1} \|u^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} + \frac{1}{2} \int_0^t \left( \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right) \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} dt' \\ &\quad \left. + \|\tau\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|u^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} + \int_0^t \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}}} dt' \right) \\ &\leq \frac{C_1}{(1-\omega)^3} \left( A^h(0) + \|\tau\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|u^d\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} + \left( \frac{(1-\omega)^3}{4C_1} + B(t) \right) \|u^h\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \right. \\ &\quad \left. + \int_0^t \left( \frac{3}{2} \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \left( \frac{C_1}{(1-\omega)^3} A_0^d + \frac{1}{2} \right) \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right) \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} dt' \right). \end{aligned}$$

In view of (3.64), we have

$$(3.72) \quad \begin{aligned} \|\tau\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|u^d\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} &\leq \frac{2C_2}{(1-\omega)^3} \left( A^d(0) + A^h(t) (A^h(t) + B(t)) \right) B(t) \\ &\leq \frac{2C_2}{(1-\omega)^3} A^d(0) B(t) + \frac{2C_2}{(1-\omega)^3} (A^h(t) + B(t)) B(t) A^h(t). \end{aligned}$$

Substituting (3.72) into (3.71), and using (3.69), one deduces that

$$(3.73) \quad \begin{aligned} A^h(t) &\leq \frac{C_1}{(1-\omega)^3} \left( A^h(0) + \frac{2C_2}{(1-\omega)^3} A^d(0) \tilde{B}_0 + \left( B(t) \left( 1 + \frac{2C_2}{(1-\omega)^3} (A^h(t) + B(t)) \right) \right) A^h(t) \right. \\ &\quad \left. + \int_0^t \left( \frac{3}{2} \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \left( \frac{C_1}{(1-\omega)^3} A_0^d + \frac{1}{2} \right) \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right) \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} dt' \right) + \frac{1}{4} A^h(t). \end{aligned}$$

If

$$(3.74) \quad B_0 \left( 1 + \frac{2C_2}{(1-\omega)^3} (A_0^h + B_0) \right) \leq \frac{(1-\omega)^3}{4C_1},$$

then (3.73) reduces to

$$(3.75) \quad \begin{aligned} A^h(t) &\leq \frac{2C_1}{(1-\omega)^3} \left( A^h(0) + \frac{2C_2}{(1-\omega)^3} A^d(0) \tilde{B}_0 \right. \\ &\quad \left. + \int_0^t \left( \frac{3}{2} \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \left( \frac{C_1}{(1-\omega)^3} A_0^d + \frac{1}{2} \right) \|u^d\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right) \|u^h\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} dt' \right). \end{aligned}$$

Gronwall's inequality implies that

$$(3.76) \quad A^h(t) \leq \frac{2C_1}{(1-\omega)^3} \exp\left(\frac{2C_1}{(1-\omega)^3} (1 + A_0^d (1 + \frac{C_1}{(1-\omega)^3} A_0^d))\right) \left( A^h(0) + \frac{2C_2}{(1-\omega)^3} A^d(0) \tilde{B}_0 \right),$$

provided

$$(3.77) \quad \frac{3}{2} A_0^h \leq 1.$$

Substituting (3.70) into (3.76), we find that

$$\begin{aligned}
 A^h(t) &\leq \frac{2C_1}{(1-\omega)^3} \exp\left(\frac{2C_1^2}{(1-\omega)^6}(1+A_0^d)^2\right) \left\{ A^h(0) \right. \\
 &\quad \left. + \frac{4C_2C_3}{(1-\omega)^3} \exp\left(2C_3(1+A_0^d)\right) (A^d(0)B(0) + \omega A^d(0)A_0^d) \right\} \\
 &\leq \frac{2C_1}{(1-\omega)^3} \exp\left(\frac{8C_1^2}{(1-\omega)^6}(A_0^d)^2\right) \left\{ A^h(0) + \frac{4C_2C_3}{(1-\omega)^3} \exp\left(4C_3A_0^d\right) A_0^d (B(0) + \omega A^d(0)) \right\} \\
 (3.78) \quad &\leq \frac{8C_1C_2C_3}{(1-\omega)^6} \exp\left(\frac{8(C_1^2+C_3)}{(1-\omega)^6}(A_0^d)^2\right) (A^h(0) + \omega A^d(0) + B(0)) = \frac{1}{2}A_0^h.
 \end{aligned}$$

Finally, substituting (3.78) into (3.69), we get the estimate of  $B(t)$  in term of  $A^h(0) + \omega A^d(0) + B(0)$ :

$$\begin{aligned}
 B(t) &\leq \frac{48C_1C_2C_3^2}{(1-\omega)^6} \exp\left(4C_3A_0^d\right) \exp\left(\frac{8(C_1^2+C_3)}{(1-\omega)^6}(A_0^d)^2\right) (2B(0) + A^h(0) + 2\omega A^d(0)) \\
 (3.79) \quad &\leq \frac{96C_1C_2^2C_3^2}{(1-\omega)^6} \exp\left(\frac{12(C_1^2+C_3)}{(1-\omega)^6}(A_0^d)^2\right) (A^h(0) + \omega A^d(0) + B(0)) = \frac{1}{2}B_0.
 \end{aligned}$$

The proof of Proposition 3.2 is completed.  $\square$

#### 4. PROOF OF THEOREM 1.1

Now we are in a position to prove the main result of this paper.

*Proof.* The local existence and uniqueness of the solution  $(u, \tau) \in \mathcal{E}_T^{\frac{d}{2}}$  to system (1.1) with initial data  $(u_0, \tau_0) \in \left(\dot{B}_{2,1}^{\frac{d}{2}-1}\right)^d \times \left(\dot{B}_{2,1}^{\frac{d}{2}}\right)^{d \times d}$  have been proved by Chemin and Masmoudi [7]. Let  $T^*$  be the lifespan of  $(u, \tau)$  obtained in [7]. Define  $T_1$  be the supremum of all time  $T' \in [0, T^*)$  such that

$$(4.1) \quad A^h(t) \leq A_0^h, \quad A^d(t) \leq A_0^d, \quad B(t) \leq B_0 \quad \text{for all } t \in [0, T'].$$

Choosing  $C_0$  satisfying (1.8) so large that (3.58)–(3.60) hold, then thanks to Proposition 3.2, for all  $0 \leq t < T_1$ , we have

$$(4.2) \quad A^h(t) \leq \frac{1}{2}A_0^h, \quad A^d(t) \leq \frac{1}{2}A_0^d, \quad B(t) \leq \frac{1}{2}B_0.$$

According to standard continuation method, we obtain  $T_1 = T^*$ . Moreover, (4.2) and the embedding  $\dot{B}_{2,1}^{\frac{d}{2}} \hookrightarrow L^\infty$  imply that

$$(4.3) \quad \int_0^{T^*} (\|\nabla u(t, \cdot)\|_{L^\infty} + \|\tau(t, \cdot)\|_{L^\infty}) dt < \infty,$$

provided (1.8) holds. Thus, the blow-up criterion in [7] enables us to conclude that  $T^* = \infty$ . This completes the proof of Theorem 1.1.  $\square$

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DEPARTMENT OF MATHEMATICS, ZHEJIANG UNIVERSITY, HANGZHOU 310027, P. R. CHINA

*E-mail address:* dyf@zju.edu.cn

SCHOOL OF MATHEMATICS AND STATISTICS, CENTRAL CHINA NORMAL UNIVERSITY, WUHAN 430079, P. R. CHINA

*E-mail address:* rzz@mail.ccnu.edu.cn